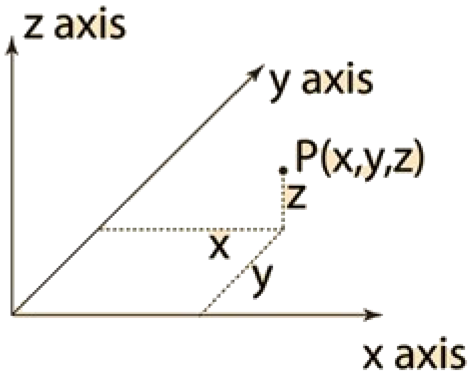
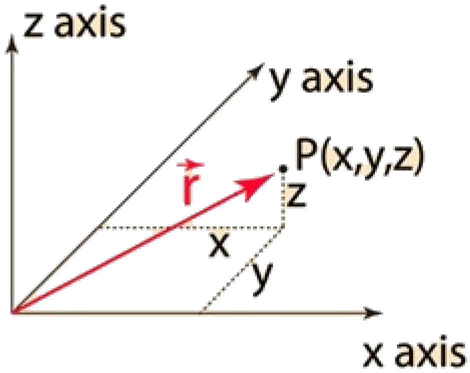
UNIT I

Coordinate Systems:
(Cartesian Coordinate System)

The most common coordinate system for representing positions in space is one based on three perpendicular s patial axes generally designated x, y, and z.



Any point P may be represented by three signed numbers, usually written (x, y, z) where the coordinate is the perpendicular distance from the plane formed by the other two axes.



Often positions are specified by a position vector r which can be expressed in terms of the coordinate values and associated unit vectors.

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

Although the entire coordinate system can be rotated, the relationship between the axes is fixed in what is called a right-handed coordinate system.

For the display of some kinds of data,it may be convenient to have different scales for the different axes, but for the purpose of mathematical operations with the coordinates, it is necessary for the axes to have the same scales. The term "Cartesian coordinates" is used to describe such systems, and the values of the three coordinates unambiguously locate a point in space. In such a coordinate system you can calculate the distance between two points and perform operations like axis rotations without altering this value.

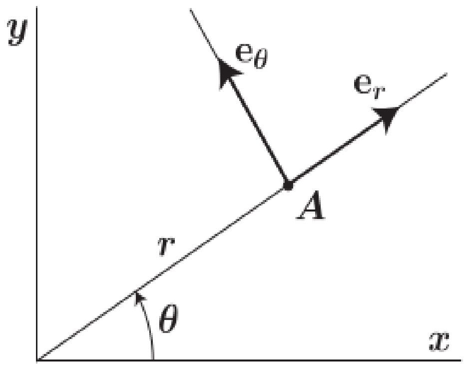
The distance between any two points in rectangular coordinates can be found from the distance relationship. In case of Cartesian Coordinate systems it is given as:

The distance between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Polar Coordinates $(r - \theta)$

In polar coordinates, the position of a particle A, is determined by the value of the radial distance to the origin, r, and the angle that the radial line makes with an arbitrary fixed line, such as the x axis. Thus, the trajectory of a particle will be determined if we know r and θ as a function of t, i.e. $r(t), \theta(t)$. The directions of increasing r and θ are defined by the orthogonal unit vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$.

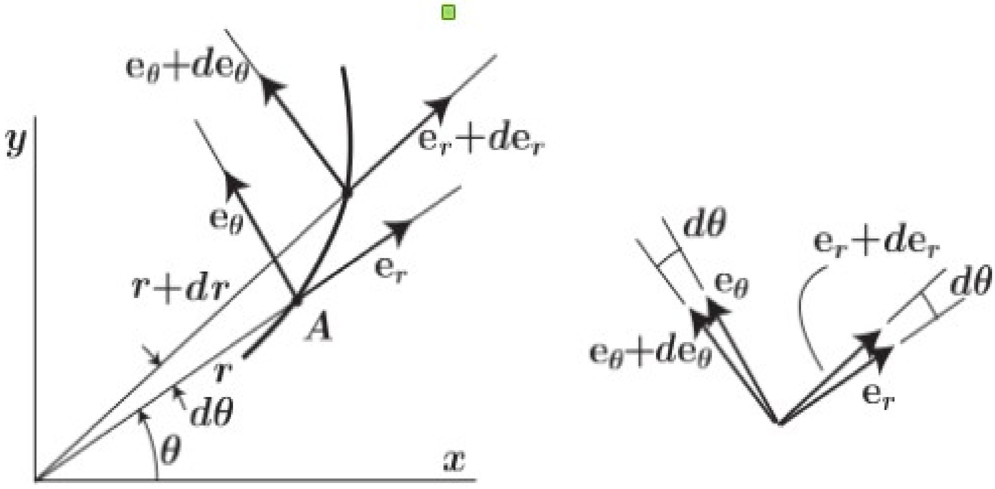


The position vector of a particle has a magnitude equal to the radial distance, and a direction determined by $\boldsymbol{e}_r$. Thus,

$$\boldsymbol{r} = r\boldsymbol{e}_r \ . \tag{1}$$

Since the vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are clearly different from point to point, their variation will have to be considered when calculating the velocity and acceleration.

Over an infinitesimal interval of time dt, the coordinates of point A will change from (r, θ), to $(r + dr, \theta + d\theta)$ as shown in the diagram.



We note that the vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ do not change when the coordinate r changes. Thus, $d\boldsymbol{e}_r/dr = \boldsymbol{0}$ and $d\boldsymbol{e}_\theta/dr = \boldsymbol{0}$. On the other hand, when θ changes to $\theta + d\theta$, the vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are rotated by an angle $d\theta$. From the diagram, we see that $d\boldsymbol{e}_r = d\theta \boldsymbol{e}_\theta$, and that $d\boldsymbol{e}_\theta = -d\theta \boldsymbol{e}_r$. This is because their magnitudes in the limit are equal to the unit vector as radius times $d\theta$ in radians. Dividing through by $d\theta$, we have,

$$\frac{d\boldsymbol{e}_r}{d\theta} = \boldsymbol{e}_\theta, \quad \text{and} \quad \frac{d\boldsymbol{e}_\theta}{d\theta} = -\boldsymbol{e}_r \ .$$

Multiplying these expressions by $d\theta/dt \equiv \dot{\theta}$, we obtain,

$$\frac{d\boldsymbol{e}_r}{d\theta}\frac{d\theta}{dt} \equiv \frac{d\boldsymbol{e}_r}{dt} = \dot{\theta}\boldsymbol{e}_\theta, \quad \text{and} \quad \frac{d\boldsymbol{e}_\theta}{dt} = -\dot{\theta}\boldsymbol{e}_r \ . \tag{2}$$

Note ***Alternative calculation of the unit vector derivatives***

An alternative, more mathematical, approach to obtaining the derivatives of the unit vectors is to express $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ in terms of their cartesian components along $\boldsymbol{i}$ and $\boldsymbol{j}$. We have that

$$\begin{aligned} \boldsymbol{e}_r &= \cos\theta \boldsymbol{i} + \sin\theta \boldsymbol{j} \\ \boldsymbol{e}_\theta &= -\sin\theta \boldsymbol{i} + \cos\theta \boldsymbol{j} \ . \end{aligned}$$

Therefore, when we differentiate we obtain,

$$\frac{d\boldsymbol{e}_r}{dr} = 0, \qquad \frac{d\boldsymbol{e}_r}{d\theta} = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j} \equiv \boldsymbol{e}_\theta$$
$$\frac{d\boldsymbol{e}_\theta}{dr} = 0, \qquad \frac{d\boldsymbol{e}_\theta}{d\theta} = -\cos\theta\boldsymbol{i} - \sin\theta\boldsymbol{j} \equiv -\boldsymbol{e}_r \ .$$

Velocity vector

We can now derive expression (1) with respect to time and write

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{r}\,\boldsymbol{e}_r + r\,\dot{\boldsymbol{e}}_r \ ,$$

or, using expression (2), we have

$$\boldsymbol{v} = \dot{r}\,\boldsymbol{e}_r + r\dot{\theta}\,\boldsymbol{e}_\theta \ . \tag{3}$$

Here, $v_r = \dot{r}$ is the *radial velocity* component, and $v_\theta = r\dot{\theta}$ is the *circumferential velocity* component. We also have that $v = \sqrt{v_r^2 + v_\theta^2}$. The radial component is the rate at which $\boldsymbol{r}$ changes magnitude, or stretches, and the circumferential component, is the rate at which $\boldsymbol{r}$ changes direction, or swings.

Acceleration vector

Differentiating again with respect to time, we obtain the acceleration

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{r}\,\boldsymbol{e}_r + \dot{r}\,\dot{\boldsymbol{e}}_r + \dot{r}\dot{\theta}\,\boldsymbol{e}_\theta + r\ddot{\theta}\,\boldsymbol{e}_\theta + r\dot{\theta}\,\dot{\boldsymbol{e}}_\theta$$

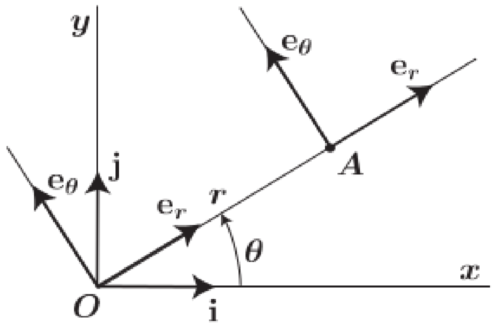
Using the expressions (2), we obtain,

$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\,\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\boldsymbol{e}_\theta \ , \tag{4}$$

where $a_r = (\ddot{r} - r\dot{\theta}^2)$ is the *radial acceleration* component, and $a_\theta = (r\ddot{\theta} + 2\dot{r}\dot{\theta})$ is the *circumferential acceleration* component. Also, we have that $a = \sqrt{a_r^2 + a_\theta^2}$.

Change of basis

In many practical situations, it will be necessary to transform the vectors expressed in polar coordinates to cartesian coordinates and vice versa.

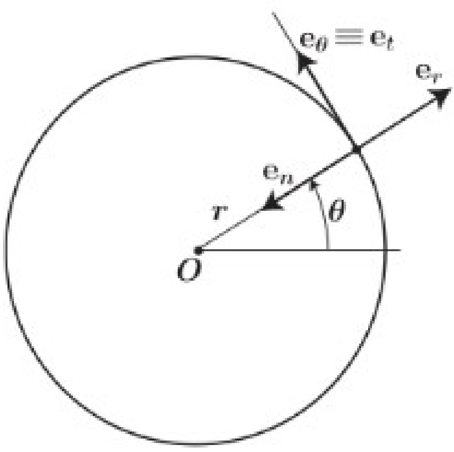


Since we are dealing with free vectors, we can translate the polar reference frame for a given point (r, θ), to the origin, and apply a standard change of basis procedure. This will give, for a generic vector $\boldsymbol{A}$,

$$\begin{pmatrix} A_r \\ A_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A_r \\ A_\theta \end{pmatrix} .$$

Example — ***Circular motion***

Consider as an illustration, the motion of a particle in a circular trajectory having angular velocity $\omega = \dot{\theta}$, and angular acceleration $\alpha = \dot{\omega}$.



In polar coordinates, the equation of the trajectory is

$$r = R = \text{constant}, \qquad \theta = \omega t + \frac{1}{2}\alpha t^2.$$

The velocity components are

$$v_r = \dot{r} = 0, \qquad \text{and} \qquad v_\theta = r\dot{\theta} = R(\omega + \alpha t) = v \ ,$$

and the acceleration components are,

$$a_r = \ddot{r} - r\dot{\theta}^2 = -R(\omega + \alpha t)^2 = -\frac{v^2}{R}, \qquad \text{and} \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = R\alpha = a_t \ ,$$

where we clearly see that, $a_r \equiv -a_n$, and that $a_\theta \equiv a_t$.

In cartesian coordinates, we have for the trajectory,

$$x = R\cos(\omega t + \frac{1}{2}\alpha t^2), \quad y = R\sin(\omega t + \frac{1}{2}\alpha t^2) \ .$$

For the velocity,

$$v_x = -R(\omega + \alpha t)\sin(\omega t + \frac{1}{2}\alpha t^2), \quad v_y = R(\omega + \alpha t)\cos(\omega t + \frac{1}{2}\alpha t^2) \ ,$$

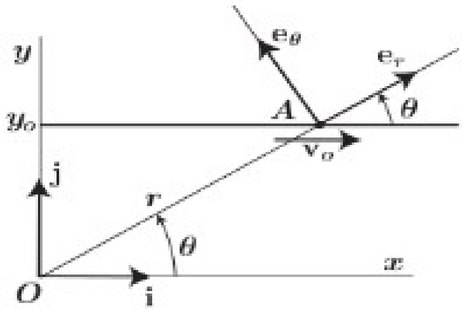
and, for the acceleration,

$$a_x = -R(\omega+\alpha t)^2\cos(\omega t+\frac{1}{2}\alpha t^2) - R\alpha\sin(\omega t+\frac{1}{2}\alpha t^2), \quad a_y = -R(\omega+\alpha t)^2\sin(\omega t+\frac{1}{2}\alpha t^2) + R\alpha\cos(\omega t+\frac{1}{2}\alpha t^2) \ .$$

We observe that, for this problem, the result is much simpler when expressed in polar (or intrinsic) coordinates.

Example — ***Motion on a straight line***

Here we consider the problem of a particle moving with constant velocity v_0, along a horizontal line $y = y_0$.



Assuming that at $t = 0$ the particle is at $x = 0$, the trajectory and velocity components in cartesian coordinates are simply,

$$\begin{aligned} x &= v_0 t & y &= y_0 \\ v_x &= v_0 & v_y &= 0 \\ a_x &= 0 & a_y &= 0 \ . \end{aligned}$$

In polar coordinates, we have,

$$r = \sqrt{v_0^2 t^2 + y_0^2} \qquad \theta = \tan^{-1}(\frac{y_0}{v_0 t})$$

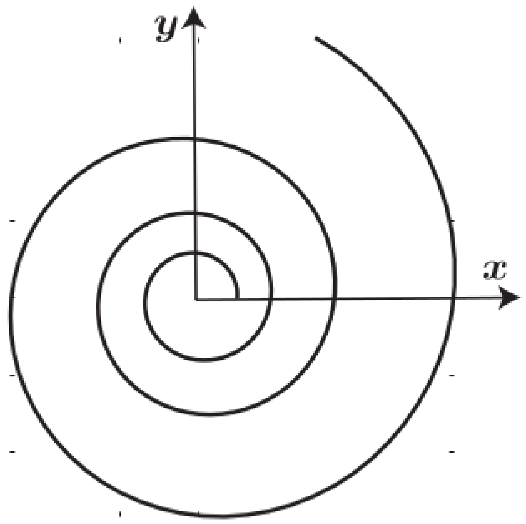
$$v_r = \dot{r} = v_0 \cos\theta \qquad v_\theta = r\dot{\theta} = -v_0 \sin\theta$$

$$a_r = \ddot{r} - r\dot{\theta}^2 = 0 \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \ \ .$$

Here, we see that the expressions obtained in cartesian coordinates are simpler than those obtained using polar coordinates. It is also reassuring that the acceleration in both the r and θ direction, calculated from the general two-term expression in polar coordinates, works out to be zero as it must for constant velocity-straight line motion.

Example ***Spiral motion (Kelppner/Kolenkow)***

A particle moves with $\dot{\theta} = \omega =$ constant and $r = r_0 e^{\beta t}$, where r_0 and β are constants.



We shall show that for certain values of β, the particle moves with $a_r = 0$.

$$\begin{aligned} \boldsymbol{a} &= (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta \\ &= (\beta^2 r_0 e^{\beta t} - r_0 e^{\beta t}\omega^2)\boldsymbol{e}_r + 2\beta r_0 \omega e^{\beta t}\boldsymbol{e}_\theta \end{aligned}$$

If $\beta = \pm\omega$, the radial part of $\boldsymbol{a}$ vanishes. It seems quite surprising that when $r = r_0 e^{\beta t}$, the particle moves with zero radial acceleration. The error is in thinking that $\ddot{r}$ makes the only contribution to a_r; the term $-r\dot{\theta}^2$ is also part of the radial acceleration, and cannot be neglected.

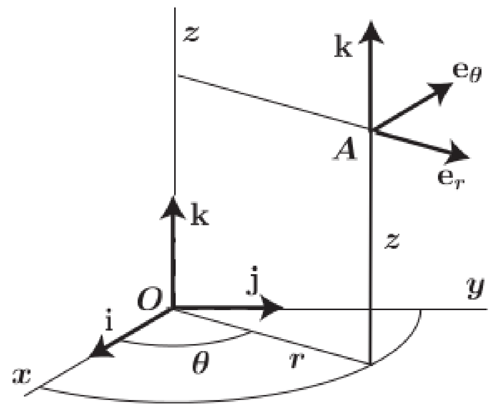
Equations of Motion

In two dimensional polar $r\theta$ coordinates, the force and acceleration vectors are $\boldsymbol{F} = F_r \boldsymbol{e}_r + F_\theta \boldsymbol{e}_\theta$ and $\boldsymbol{a} = a_r \boldsymbol{e}_r + a_\theta \boldsymbol{e}_\theta$. Thus, in component form, we have,

$$\begin{aligned} F_r &= m\, a_r = m\,(\ddot{r} - r\dot{\theta}^2) \\ F_\theta &= m\, a_\theta = m\,(r\ddot{\theta} + 2\dot{r}\dot{\theta})\ . \end{aligned}$$

Cylindrical Coordinates $(r - \theta - z)$

Polar coordinates can be extended to three dimensions in a very straightforward manner. We simply add the z coordinate, which is then treated in a cartesian like manner. Every point in space is determined by the r and θ coordinates of its projection in the xy plane, and its z coordinate.



The unit vectors $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$ and $\boldsymbol{k}$, expressed in cartesian coordinates, are,

$$\begin{aligned} \boldsymbol{e}_r &= \cos\theta \boldsymbol{i} + \sin\theta \boldsymbol{j} \\ \boldsymbol{e}_\theta &= -\sin\theta \boldsymbol{i} + \cos\theta \boldsymbol{j} \end{aligned}$$

and their derivatives,

$$\dot{\boldsymbol{e}}_r = \dot{\theta}\boldsymbol{e}_\theta, \quad \dot{\boldsymbol{e}}_\theta = -\dot{\theta}\boldsymbol{e}_r, \quad \dot{\boldsymbol{k}} = \boldsymbol{0}\ .$$

The kinematic vectors can now be expressed relative to the unit vectors $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$ and $\boldsymbol{k}$. Thus, the position vector is

$$\boldsymbol{r} = r\,\boldsymbol{e}_r + z\,\boldsymbol{k}\ ,$$

and the velocity,

$$\boldsymbol{v} = \dot{r}\,\boldsymbol{e}_r + r\dot{\theta}\,\boldsymbol{e}_\theta + \dot{z}\,\boldsymbol{k}\ ,$$

where $v_r = \dot{r}$, $v_\theta = r\dot{\theta}$, $v_z = \dot{z}$, and $v = \sqrt{v_r^2 + v_\theta^2 + v_z^2}$. Finally, the acceleration becomes

$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\,\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\boldsymbol{e}_\theta + \ddot{z}\boldsymbol{k}\ ,$$

where $a_r = \ddot{r} - r\dot{\theta}^2$, $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$, $a_z = \ddot{z}$, and $a = \sqrt{a_r^2 + a_\theta^2 + a_z^2}$.

Note that when using cylindrical coordinates, r is not the modulus of $\boldsymbol{r}$. This is somewhat confusing, but it is consistent with the notation used by most books. Whenever we use cylindrical coordinates, we will write $|\boldsymbol{r}|$ explicitly, to indicate the modulus of $\boldsymbol{r}$, i.e. $|\boldsymbol{r}| = \sqrt{r^2 + z^2}$.

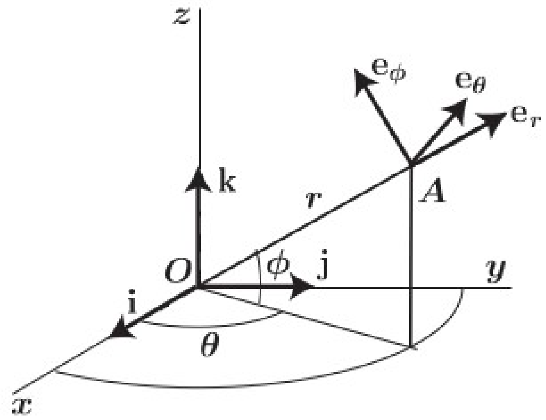
Equations of Motion

In cylindrical $r\theta z$ coordinates, the force and acceleration vectors are $\boldsymbol{F} = F_r \boldsymbol{e}_r + F_\theta \boldsymbol{e}_\theta + F_z \boldsymbol{e}_z$ and $\boldsymbol{a} = a_r \boldsymbol{e}_r + a_\theta \boldsymbol{e}_\theta + a_z \boldsymbol{e}_z$. Thus, in component form we have,

$$\begin{aligned} F_r &= m\,a_r = m\,(\ddot{r} - r\dot{\theta}^2) \\ F_\theta &= m\,a_\theta = m\,(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \\ F_z &= m\,a_z = m\,\ddot{z}\ . \end{aligned}$$

Spherical Coordinates $(r - \theta - \phi)$

In spherical coordinates, we utilize two angles and a distance to specify the position of a particle, as in the case of radar measurements, for example.



and for the kinematic vectors

$$
\begin{aligned}
\boldsymbol{r} &= r\,\boldsymbol{e}_r \\
\boldsymbol{v} &= \dot{r}\boldsymbol{e}_r + r\dot{\theta}\cos\phi\,\boldsymbol{e}_\theta + r\dot{\phi}\,\boldsymbol{e}_\phi \\
\boldsymbol{a} &= (\ddot{r} - r\dot{\theta}^2\cos^2\phi - r\dot{\phi}^2)\,\boldsymbol{e}_r \\
&+ (2\dot{r}\dot{\theta}\cos\phi + r\ddot{\theta}\cos\phi - 2r\dot{\theta}\dot{\phi}\sin\phi)\,\boldsymbol{e}_\theta \\
&+ (2\dot{r}\dot{\phi} + r\dot{\phi}^2\sin\phi\,\cos\phi + r\ddot{\phi})\,\boldsymbol{e}_\phi\ .
\end{aligned}
$$

Equations of Motion

Finally, in spherical $r\theta\phi$ coordinates, we write $\boldsymbol{F} = F_r\boldsymbol{e}_r + F_\theta\boldsymbol{e}_\theta + F_\phi\boldsymbol{e}_\phi$ and $\boldsymbol{a} = a_r\boldsymbol{e}_r + a_\theta\boldsymbol{e}_\theta + a_\phi\boldsymbol{e}_\phi$. Thus,

$$
\begin{aligned}
F_r &= m\,a_r = m\,(\ddot{r} - r\dot{\theta}^2\cos^2\phi - r\dot{\phi}^2) \\
F_\theta &= m\,a_\theta = m\,(2\dot{r}\dot{\theta}\cos\phi + r\ddot{\theta}\cos\phi - 2r\dot{\theta}\dot{\phi}\sin\phi) \\
F_\phi &= m\,a_\phi = m\,(2\dot{r}\dot{\phi} + r\dot{\phi}^2\sin\phi\cos\phi + r\ddot{\phi})\ .
\end{aligned}
$$

Inertial and Non-Inertial Reference Frame:

A frame of reference in which Newton's laws hold is called an inertial frame of reference. Any frame which is at rest or is moving with constant velocity with respect to inertial frame is also an inertial frame of reference.

Frame of reference in which Newton's law dose not hold is called non-inertial frame of reference. An accelerated frame of reference is an example of non-inertial frame of reference.

Rotating Frames:

In this section we will discuss what Newton's equations of motion look like in non-inertial frames. Just as there are many ways that an animal can not be a dog, so there are many ways in which a reference frame can be non-inertial. Here we will just consider one type: reference frames that rotate

Let's start with the inertial frame S drawn in the figure with coordinate axes x, y and z. Our goal is to understand the motion of particles as seen in a non-inertial frame S', with axes x', y' and z', which is rotating with respect to S. We'll denote the angle between the x-axis of S and the x'-axis of S' as θ. Since S' is rotating, we clearly have $\theta = \theta(t)$ and $\dot{\theta} \neq 0$.

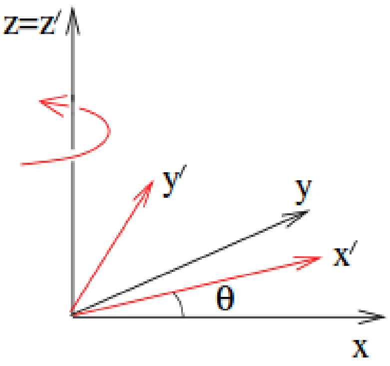


Figure 31:

Our first task is to find a way to describe the rotation of the axes. For this, we can use the angular velocity vector $\boldsymbol{\omega}$ that we introduced in the last section to describe the motion of particles. Consider a particle that is sitting stationary in the S' frame. Then, from the perspective of frame S it will appear to be moving with velocity

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$$

where, in the present case, $\boldsymbol{\omega} = \dot{\theta}\hat{\mathbf{z}}$. Recall that in general, $|\boldsymbol{\omega}| = \dot{\theta}$ is the angular speed, while the direction of $\boldsymbol{\omega}$ is the axis of rotation, defined in a right-handed sense.

We can extend this description of the rotation of the axes of S' themselves. Let $\mathbf{e}'_i$, $i = 1, 2, 3$ be the unit vectors that point along the x', y' and z' directions of S'. Then these also rotate with velocity

Velocity and Acceleration in a Rotating Frame

Consider now a particle which is no longer stuck in the S' frame, but moves on some trajectory. We can measure the position of the particle in the inertial frame S, where, using the summation convention, we write

$$\mathbf{r} = r_i \mathbf{e}_i$$

Here the unit vectors $\mathbf{e}_i$, with $i = 1, 2, 3$ point along the axes of S. Alternatively, we can measure the position of the particle in frame S', where the position is

$$\mathbf{r} = r'_i \mathbf{e}'_i$$

Note that the position vector $\mathbf{r}$ is the same in both of these expressions: but the coordinates r_i and r'_i differ because they are measured with respect to different axes. Now, we can compute an expression for the velocity of the particle. In frame S, it is simply

$$\dot{\mathbf{r}} = \dot{r}_i \mathbf{e}_i \tag{6.1}$$

because the axes $\mathbf{e}_i$ do not change with time. However, in the rotating frame S', the velocity of the particle is

$$\begin{aligned}\dot{\mathbf{r}} &= \dot{r}'_i \mathbf{e}'_i + r'_i \dot{\mathbf{e}}'_i \\ &= \dot{r}'_i \mathbf{e}'_i + r'_i \boldsymbol{\omega} \times \mathbf{e}'_i \\ &= \dot{r}'_i \mathbf{e}'_i + \boldsymbol{\omega} \times \mathbf{r}\end{aligned} \tag{6.2}$$

$$\left(\frac{d\mathbf{r}}{dt}\right)_S = \left(\frac{d\mathbf{r}}{dt}\right)_{S'} + \boldsymbol{\omega} \times \mathbf{r} \tag{6.3}$$

We'll introduce a slightly novel notation to help highlight the physics hiding in these two equations. We write the velocity of the particle as seen by an observer in frame S as

$$\left(\frac{d\mathbf{r}}{dt}\right)_S = \dot{r}_i \mathbf{e}_i$$

Similarly, the velocity as seen by an observer in frame S' is just

$$\left(\frac{d\mathbf{r}}{dt}\right)_{S'} = \dot{r}'_i \mathbf{e}'_i$$

From equations (6.1) and (6.2), we see that the two observers measure different velocities,

What about acceleration? We can play the same game. In frame S, we have

$$\ddot{\mathbf{r}} = \ddot{r}_i \mathbf{e}_i$$

while in frame S', the expression is a little more complicated. Differentiating (6.2) once more, we have

$$\begin{aligned}\ddot{\mathbf{r}} &= \ddot{r}'_i \mathbf{e}'_i + \dot{r}'_i \dot{\mathbf{e}}_i{}' + \dot{r}'_i \boldsymbol{\omega} \times \mathbf{e}'_i + r'_i \dot{\boldsymbol{\omega}} \times \mathbf{e}'_i + r'_i \boldsymbol{\omega} \times \dot{\mathbf{e}}_i{}' \\ &= \ddot{r}'_i \mathbf{e}'_i + 2\dot{r}'_i \boldsymbol{\omega} \times \mathbf{e}'_i + \dot{\boldsymbol{\omega}} \times \mathbf{r} + r'_i \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{e}_i{}')\end{aligned}$$

As with velocities, the acceleration seen by the observer in S is $\ddot{r}_i\mathbf{e}_i$ while the acceleration seen by the observer in S' is $\ddot{r}'_i\mathbf{e}'_i$. Equating the two equations above gives us

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)_S = \left(\frac{d^2\mathbf{r}}{dt^2}\right)_{S'} + 2\omega \times \left(\frac{d\mathbf{r}}{dt}\right)_{S'} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{6.40}$$

This equation contains the key to understanding the motion of particles in a rotating frame.

Newton's Equation of Motion in a Rotating Frame

With the hard work behind us, let's see how a person sitting in the rotating frame S' would see Newton's law of motion. We know that in the inertial frame S, we have

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_S = \mathbf{F}$$

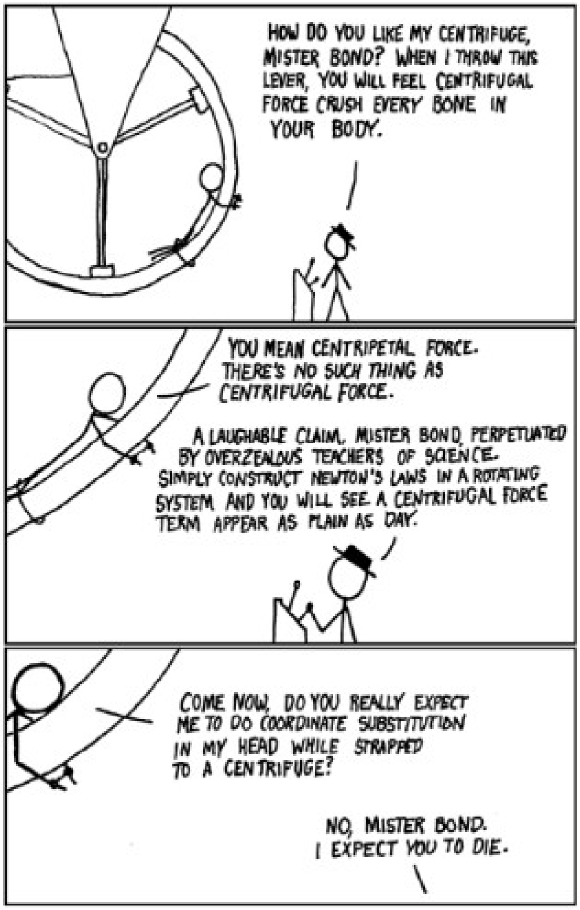
So, using (6.4), in frame S', we have

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{S'} = \mathbf{F} - 2m\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{S'} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{6.5}$$

In other words, to explain the motion of a particle an observer in S' must invoke the existence of three further terms on the right-hand side of Newton's equation. These are called *fictitious forces*. Viewed from S', a free particle doesn't travel in a straight line and these fictitious forces are necessary to explain this departure from uniform motion. In the rest of this section, we will see several examples of this.

The $-2m\boldsymbol{\omega} \times \dot{\mathbf{r}}$ term in (6.5) is the *Coriolis force*; the $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ term is called the *centrifugal force*; the $-m\dot{\boldsymbol{\omega}} \times \mathbf{r}$ term is called the *Euler force*.

The most familiar non-inertial frame is the room you are sitting in. It rotates once per day around the north-south axis of the Earth. It further rotates once a year about the Sun which, in turn, rotates about the centre of the galaxy. From these time scales, we can easily compute $\omega = |\boldsymbol{\omega}|$.



The radius of the Earth is $R_{\text{Earth}} \approx 6 \times 10^3\ km$. The Earth rotates with angular frequency

$$\omega_{\text{rot}} = \frac{2\pi}{1\ \text{day}} \approx 7 \times 10^{-5}\ s^{-1}$$

The distance from the Earth to the Sun is $a_e \approx 2 \times 10^8\ km$. The angular frequency of the orbit is

$$\omega_{\text{orb}} = \frac{2\pi}{1\ \text{year}} \approx 2 \times 10^{-7}\ s^{-1}$$

It should come as no surprise to learn that $\omega_{\text{rot}}/\omega_{\text{orb}} = T_{\text{orb}}/T_{\text{rot}} \approx 365$.

In what follows, we will see the effect of the centrifugal and Coriolis forces on our daily lives. We will not discuss the Euler force, which arises only when the speed of the rotation changes with time. Although this plays a role in various funfair rides, it's not important in the frame of the Earth. (The angular velocity of the Earth's rotation does, in fact, have a small, but non-vanishing, $\dot{\boldsymbol{\omega}}$ due to the precession and nutation of the Earth's rotational axis. However, it is tiny, with $\dot{\omega} \ll \omega^2$ and, as far as I know, the resulting Euler force has no consequence).

Inertial vs Gravitational Mass Revisited

Notice that all the fictitious forces are proportional to the inertial mass m. There is no mystery here: it's because they all originated from the "ma" side of "F=ma" rather than "F" side. But, as we mentioned in Section 2, experimentally the gravitational force also appears to be proportional to the inertial mass. Is this evidence that gravity too is a fictitious force? In fact it is. Einstein's theory of general relativity recasts gravity as the fictitious force that we experience due to the curvature of space and time.

The centrifugal force is given by

$$\begin{aligned}\mathbf{F}_{\text{cent}} &= -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \\ &= -m(\boldsymbol{\omega} \cdot \mathbf{r})\,\boldsymbol{\omega} + m\omega^2 \mathbf{r}\end{aligned}$$

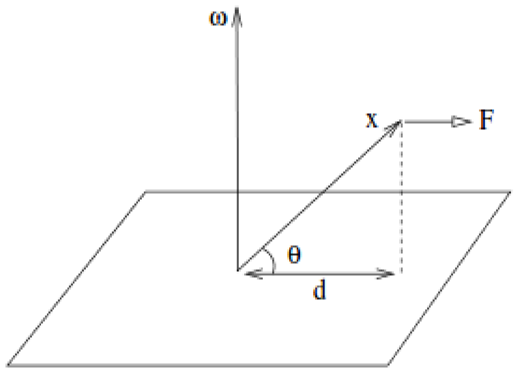


Figure 33:

We can get a feel for this by looking at the figure. The vector $\boldsymbol{\omega} \times \mathbf{r}$ points into the page, which means that $-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ points away from the axis of rotation as shown. The magnitude of the force is

$$|\mathbf{F}_{\text{cent}}| = m\omega^2 r \cos\theta = m\omega^2 d \qquad (6.6)$$

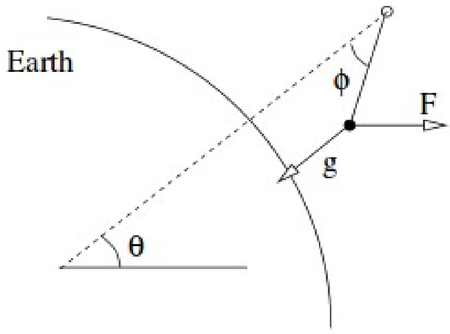
where d is the distance to the axis of rotation as shown in the figure.

The centrifugal force does not depend on the velocity of the particle. In fact, it is an example of a conservative force. We can see this by writing

$$\mathbf{F}_{\text{cent}} = -\nabla V \quad \text{with} \quad V = -\frac{m}{2}(\boldsymbol{\omega} \times \mathbf{r})^2 \qquad (6.7)$$

In a rotating frame, V has the interpretation of the potential energy associated to a particle. The potential V is negative, which tells us that particles want to fly out from the axis of rotation to lower their energy by increasing $|\mathbf{r}|$.

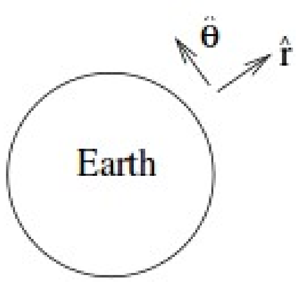
Suspend a piece of string from the ceiling. You might expect that the string points down to the centre of the Earth. But the effect of the centrifugal force due to the Earth's rotation means that this isn't the case. A somewhat exaggerated picture of this is shown in the figure. The question that we would like to answer is: what is the angle ϕ that the string makes with the line pointing to the Earth's centre? As we will now show, the angle ϕ depends on the latitude, θ, at which we're sitting.



The effective acceleration, due to the combination of gravity and the centrifugal force is

$$\mathbf{g}_{\text{eff}} = \mathbf{g} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

It is useful to resolve this acceleration in the radial and southerly directions by using the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. The centrifugal force $\mathbf{F}$ is resolved as



$$\begin{aligned}\mathbf{F} &= |\mathbf{F}|\cos\theta\,\hat{\mathbf{r}} - |\mathbf{F}|\sin\theta\,\hat{\boldsymbol{\theta}} \\ &= m\omega^2 r\cos^2\theta\,\hat{\mathbf{r}} - m\omega^2 r\cos\theta\sin\theta\,\hat{\boldsymbol{\theta}}\end{aligned}$$

where, in the second line, we have used the magnitude of the centrifugal force computed in (6.6). Notice that, at the pole $\theta = \pi/2$ and the centrifugal forces vanishes as expected. This gives the effective acceleration

$$\mathbf{g}_{\text{eff}} = -g\hat{\mathbf{r}} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = (-g + \omega^2 R\cos^2\theta)\hat{\mathbf{r}} - \omega^2 R\cos\theta\sin\theta\,\hat{\boldsymbol{\theta}}$$

where R is the radius of the Earth.

The force $m\mathbf{g}_{\text{eff}}$ must be balanced by the tension $\mathbf{T}$ in the string. This too can be resolved as

$$\mathbf{T} = T\cos\phi\,\hat{\mathbf{r}} + T\sin\phi\,\hat{\boldsymbol{\theta}}$$

In equilibrium, we need $m\mathbf{g}_{\text{eff}} + \mathbf{T} = 0$, which allows us to eliminate T to get an equation relating ϕ to the latitude θ,

$$\tan\phi = \frac{\omega^2 R\cos\theta\sin\theta}{g - \omega^2 R\cos^2\theta}$$

This is the answer we wanted. Let's see at what latitude the angle ϕ is largest. If we compute $d(\tan\phi)/d\theta$, we find a fairly complicated expression. However, if we take int account the fact that $\omega^2 R \approx 3\times 10^{-2}\ ms^{-2} \ll g$ then we can neglect the term in which we differentiate the denominator. We learn that the maximum departure from the vertical occurs more or less when $d(\cos\theta\sin\theta)/d\theta = 0$. Or, in other words, at a latitude of $\theta \approx 45°$. However, even at this point the deflection from the vertical is tiny: an order of magnitude gives $\phi \approx 10^{-4}$.

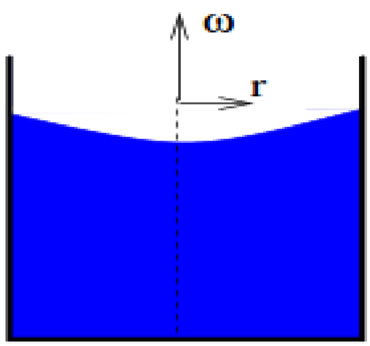
When we sit at the equator, with $\theta = 0$, then $\phi = 0$ and the string hangs directly towards the centre of the Earth. However, gravity is somewhat weaker due to the centrifugal force. We have

$$g_{\text{eff}}|_{\text{equator}} = g - \omega^2 R$$

Based on this, we expect $g_{\text{eff}} - g \approx 3\times 10^{-2}\ ms^{-2}$ at the equator. In fact, the experimental result is more like $5\times 10^{-2}\ ms^{-2}$. The reason for this discrepancy can also be traced to the centrifugal force which means that the Earth is not spherical, but rather bulges near the equator.

A Rotating Bucket

Fill a bucket with water and spin it. The surface of the water will form a concave shape like that shown in the figure. What is the shape?



We assume that the water spins with the bucket. The potential energy of a water molecule then has two contributions: one from gravity and the other due to the centrifugal force given in (6.7)

$$V_{\text{water}} = mgz - \frac{1}{2}m\omega^2 r^2$$

Now we use a somewhat slick physics argument. Consider a water molecule on the surface of the fluid. If it could lower its energy by moving along the surface, then it would. But we're looking for the equilibrium shape of the surface, which means that each point on the surface must have equal potential energy. This means that the shape of the surface is a parabola, governed by the equation

$$z = \frac{\omega^2 r^2}{2g} + \text{constant}$$

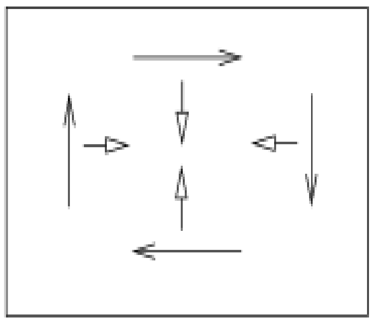
Coriolis Force

The Coriolis force is given by

$$\mathbf{F}_{\text{cor}} = -2m\boldsymbol{\omega} \times \mathbf{v}$$

where, from (6.5), we see that $\mathbf{v} = (d\mathbf{r}/dt)_{S'}$ is the velocity of the particle measured in the rotating frame S'. The force is velocity dependent: it is only felt by moving particles. Moreover, it is independent on the position.

Particles, Baths and Hurricanes

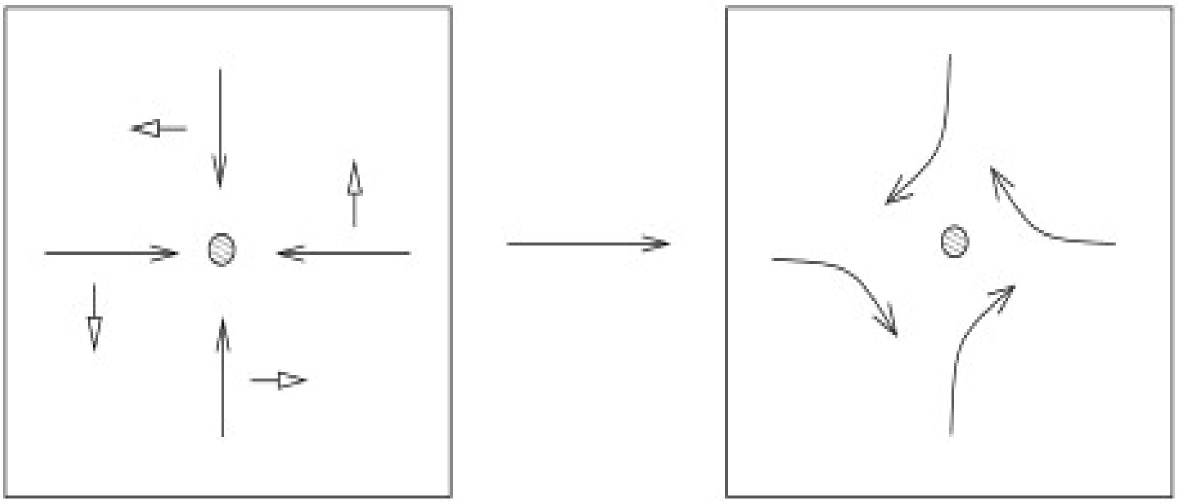


The mathematical form of the Coriolis force is identical to the Lorentz force describing a particle moving in a magnetic field. This means we already know what the effect of the Coriolis force will be: it makes moving particles turn in circles.

We can easily check that this is indeed the case. Consider a particle moving on a spinning plane as shown in the figure, where $\boldsymbol{\omega}$ is coming out of the page. In the diagram we have drawn various particle velocities, together with the Coriolis force experienced by the particle. We see that the effect of the Coriolis force is that a free particle travelling on the plane will move in a clockwise direction.

There is a similar force — at least in principle — when you pull the plug from your bathroom sink. But here there's a subtle difference which actually reverses the direction of motion!

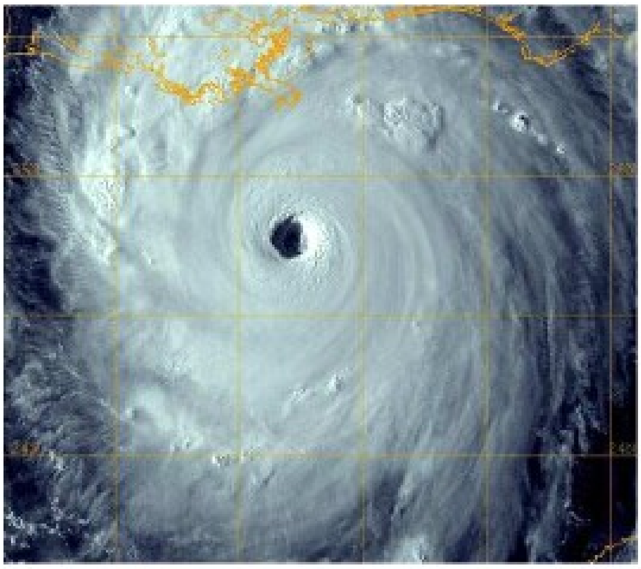
Consider a fluid in which there is a region of low pressure. This region could be formed in a sink because we pulled the plug, or it could be formed in the atmosphere due to random weather fluctuations. Now the particles in the fluid will move radially towards the low pressure region. As they move, they will be deflected by the Coriolis force as shown in the figure. The direction of the deflection is the same as that of a particle moving in the plane. But the net effect is that the swirling fluid moves in an anti-clockwise direction.



The Coriolis force is responsible for the formation of hurricanes. These rotate in an anti-clockwise direction in the Northern hemisphere and a clockwise direction in

Cyclone Catarina which hit Brazil in 2004

Hurricane Katrina, which hit New Orleans in 2005



the Southern hemisphere. However, don't spend too long staring at the rotation in your bath water. Although the effect can be reproduced in laboratory settings, in your bathroom the Coriolis force is too small: it is no more likely to make your bath water change direction than it is to make your CD change direction. (An aside: If you've not come across a CD before, you should think of them as an old fashioned ipod. There are a couple of museums in town – Fopp and HMV – which display examples of CD cases for people to look at).

Our discussion above supposed that objects were moving on a plane which is perpendicular to the angular velocity $\boldsymbol{\omega}$. But that's not true for hurricanes: they move along the surface of the Earth, which means that their velocity has a component parallel to $\boldsymbol{\omega}$. In this case, the effective magnitude of the Coriolis force gets a geometric factor,

$$|\mathbf{F}_{\rm cor}| = 2m\omega v \sin\theta \tag{6.8}$$

It's simplest to see the $\sin\theta$ factor in the case of a particle travelling North. Here the Coriolis force acts in an Easterly direction and a little bit of trigonometry shows that the force has magnitude $2m\omega v\sin\theta$ as claimed. This is particularly clear at the equator where $\theta = 0$. Here a particle travelling North has $\mathbf{v}$ parallel to $\boldsymbol{\omega}$ and so the Coriolis force vanishes.

It's a little more tricky to see the $\sin\theta$ factor for a particle travelling in the Easterly direction. In this case, $\mathbf{v}$ is perpendicular to $\boldsymbol{\omega}$, so the magnitude of the force is actually $2m\omega v$, with no trigonometric factor. However, the direction of the force no longer lies parallel to the Earth's surface: it has a component which points directly upwards. But we're not interested in this component; it's certainly not going to be big enough to compete with gravity. Projecting onto the component that lies parallel to the Earth's surface (in a Southerly direction in this case), we again get a $\sin\theta$ factor.

Conservation Laws for Systems of Particles

It is important to note that the center of mass is a property of the system and does not depend on the reference frame used. In particular, if we change the location of the origin O, $\boldsymbol{r}_G$ will change, but the absolute position of the point G within the system will not. Often, it will be convenient to describe the motion of particle i as the motion of G plus the motion of i relative to G. To this end, we introduce the relative position vector, $\boldsymbol{r}'_i$, and write,

$$\boldsymbol{r}_i = \boldsymbol{r}_G + \boldsymbol{r}'_i \; . \tag{2}$$

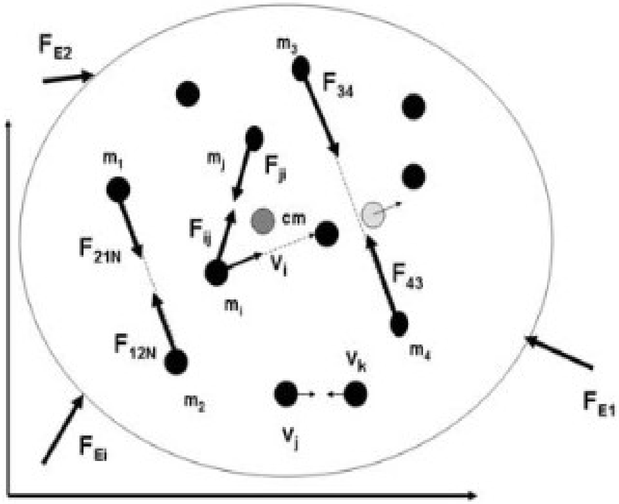
It follows immediately, from the definition of the center of mass (1) and the definition of the relative vector $\boldsymbol{r}'_i$ (2), that,

$$\sum_{i=1}^{n} m_i \boldsymbol{r}'_i = \sum_{i=1}^{n} m_i (\boldsymbol{r}_i - \boldsymbol{r}_G) = \sum_{i=1}^{n} m_i \boldsymbol{r}_i - m \boldsymbol{r}_G = 0 \; . \tag{3}$$

This result will simplify our later analysis.

Forces

In order to derive conservation laws for our system, we isolate it a little more carefully, identify what mass particles it contains and what forces act upon the individual particles.



We will consider two types of forces acting on the particles :

Conservation of Linear Momentum

The linear momentum of the system is defined as,

$$\boldsymbol{L} = \sum_{i=1}^{n} m_i \boldsymbol{v}_i \ . \tag{4}$$

From equation (2), we have that $\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i = \dot{\boldsymbol{r}}_G + \dot{\boldsymbol{r}}'_i$, which, combined with the above equation, gives,

$$\boldsymbol{L} = \sum_{i=1}^{n} m_i(\dot{\boldsymbol{r}}_G + \dot{\boldsymbol{r}}'_i) = \sum_{i=1}^{n} m_i \boldsymbol{v}_G + \frac{d}{dt}(\sum_{i=1}^{n} m_i \boldsymbol{r}'_i) = m\boldsymbol{v}_G \ , \tag{5}$$

since by the definition of center of mass $\sum_{i=1}^{n} m_i \boldsymbol{r}'_i = \boldsymbol{0}$. We now consider the time variation of the linear momentum. If we assume that the reference frame xyz is inertial, then, starting from equation (4), we have,

$$\dot{\boldsymbol{L}} = \sum_{i=1}^{n} m_i \boldsymbol{a}_i = \sum_{i=1}^{n}(\boldsymbol{F}_i + \sum_{j=1}^{n} \sum_{j\neq i} \boldsymbol{f}_{ij}) = \sum_{i=1}^{n} \boldsymbol{F}_i = \boldsymbol{F}, \tag{6}$$

where $\boldsymbol{F}$ is the sum of all external forces acting on the system. Since the sum of the internal forces balance when summed over $\boldsymbol{i}$ and $\boldsymbol{j}$, we are left with only the summation over the external forces. Thus, for a system of particles, we have that,

$$\dot{\boldsymbol{L}} = \boldsymbol{F} \ . \tag{7}$$

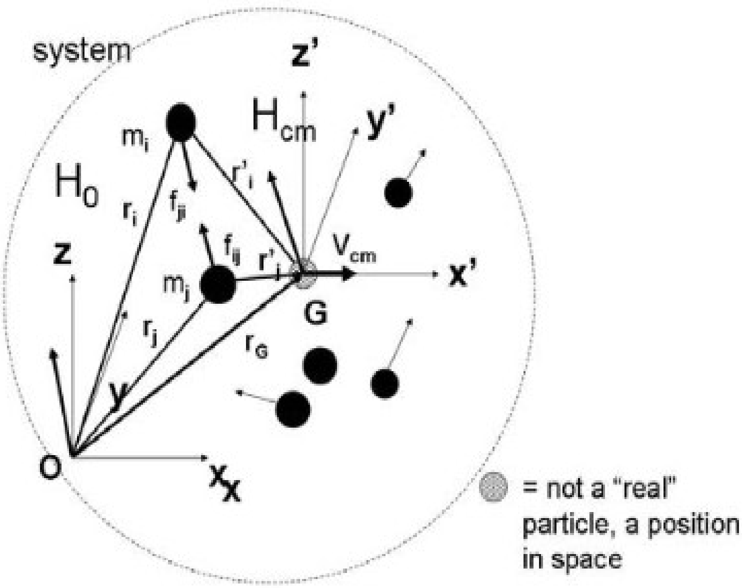
Note that, from equation (5), we can also write $\dot{\boldsymbol{L}} = m\boldsymbol{a}_G$. This is a powerful result. Note that the center of mass is in general not fixed to a particular particle but is a point in space about which the individual particles move.

These ideas also describe the conservation of linear momentum under external and internal collisions. Since individual internal collisions between particles in the system conserve momentum, the sum of their interactions also conserves momentum. If we consider an external particle imparting momentum to the system, it could be treated as an external impulse. Conversely we can consider the particle about to collide to be a part of the system, and include its momentum as part of total system momentum, which is then conserved by Newton's law.

Conservation of Angular Momentum

Since the angular momentum is defined with respect to a point in space, we will consider two cases, using a different reference point for each case: 1) conservation of angular momentum about a fixed (or more

Conservation of Angular Momentum about a Fixed Point O



The angular momentum of a system of particles about a fixed point, O, is the sum of the angular momentum of the individual particles,

$$\boldsymbol{H}_O = \sum_{i=1}^{n} (\boldsymbol{r}_i \times m_i \boldsymbol{v}_i) \ . \tag{8}$$

The time variation of $\boldsymbol{H}_O$ can be written as,

$$\dot{\boldsymbol{H}}_O = \sum_{i=1}^{n}(\dot{\boldsymbol{r}}_i \times m_i \boldsymbol{v}_i) + \sum_{i=1}^{n}(\boldsymbol{r}_i \times m_i \dot{\boldsymbol{v}}_i) = \boldsymbol{0} + \sum_{i=1}^{n}(\boldsymbol{r}_i \times (\boldsymbol{F}_i + \sum_{j=1,\ j\neq i} \boldsymbol{f}_{ij})) = \sum_{i=1}^{n}(\boldsymbol{r}_i \times \boldsymbol{F}_i) + \sum_{i=1}^{n} M_i \ . \quad (9)$$

where we replace $m\dot{\boldsymbol{v}}_i$ by the sum of the forces acting on particle **i**: $m\dot{\boldsymbol{v}}_i = (\boldsymbol{F}_i + \sum_{j=1,\ j\neq i} \cdot \boldsymbol{f}_{ij})$. $\sum_{i=1}^{n} M_i$ is the sum of any external moments that act on the system. The term $(\dot{\boldsymbol{r}}_i \times m_i \boldsymbol{v}_i)$ in equation (9) is zero; since $\dot{\boldsymbol{r}}_i = \boldsymbol{v}_i$, the two vector are parallel and their cross-product is zero. In the second term we may write $\boldsymbol{r}_i \times \boldsymbol{f}_{ij} + \boldsymbol{r}_j \times \boldsymbol{f}_{ji} = (\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{f}_{ij} = \boldsymbol{0}$ since the forces are aligned with $(\boldsymbol{r}_i$-$\boldsymbol{r}_j)$, and their values are equal and opposite, their cross product with $(\boldsymbol{r}_i$-$\boldsymbol{r}_j)$ is zero, and therefore, the internal forces have no net effect on the total angular momentum change of the particle system. Therefore

$$\dot{\boldsymbol{H}}_O = \sum_{i=1}^{n}(\boldsymbol{r}_i \times m_i \dot{\boldsymbol{v}}_i) = \sum_{i=1}^{n}(\boldsymbol{r}_i \times \boldsymbol{F}_i) + \sum_{i=1}^{n} M_i \ . \quad (10)$$

When evaluating the moments which act to change the angular momentum from equation (13), we see that the sum of the internal moments is zero so that the only moment which acts to change the angular momentum is the moments created by the external forces about the point O plus any external moments applied to the system.

Thus, we have that

$$\dot{\boldsymbol{H}}_O = \boldsymbol{M}_O \ , \quad (11)$$

where $\boldsymbol{M}_O = \sum_{i=1}^{n}(\boldsymbol{r}_i \times \boldsymbol{F}_i) + \sum_{i=1}^{n} M_i$ is the total moment, about O, due to the applied external forces plus any external moments.

Conservation of Angular Momentum about G

The angular momentum about the center of mass G is given by,

$$\boldsymbol{H}_G = \sum_{i=1}^{n} (\boldsymbol{r}'_i \times m_i \boldsymbol{v}_i) \ . \tag{12}$$

Taking the time derivative of equation (2), we obtain

$$\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i = \dot{\boldsymbol{r}}_G + \dot{\boldsymbol{r}}'_i = \boldsymbol{v}_G + \boldsymbol{v}'_i \ . \tag{13}$$

Inserting this expression into equation 12, we obtain

$$\boldsymbol{H}_G = \sum_{i=1}^{n} (\boldsymbol{r}'_i \times m_i(\dot{\boldsymbol{r}}_G + \dot{\boldsymbol{r}}'_i)) = \sum_{i=1}^{n} (\boldsymbol{r}'_i \times m_i \dot{\boldsymbol{r}}_G) + \sum_{i=1}^{n} (\boldsymbol{r}'_i \times m_i \dot{\boldsymbol{r}}'_i) = \sum_{i=1}^{n} (\boldsymbol{r}'_i \times m_i \boldsymbol{v}'_i) \ , \tag{14}$$

since $\sum_{i=1}^{n} (\boldsymbol{r}'_i \times m_i \dot{\boldsymbol{r}}_G) = -\dot{\boldsymbol{r}}_G \times \sum_{i=1}^{n} m_i \boldsymbol{r}'_i = \boldsymbol{0}$ (see equation 3). We note that equations (12) and (14) give us alternative representations for $\boldsymbol{H}_G$. Equation (12) is called the *absolute* angular momentum (since it involves absolute velocities, $\boldsymbol{v}_i$), whereas equation (14) is called the *relative* angular momentum (since it involves velocities, $\boldsymbol{v}'_i$, relative to G). When G is chosen to be the origin for the relative velocities, both the absolute and relative angular momentum are identical. In general, the absolute and relative angular momentum with respect to an ***arbitrary point*** are not the same.

We can now go back to equation (12) and consider the time variation of $\boldsymbol{H}_G$,

$$\dot{\boldsymbol{H}}_G \ = \ \sum_{i=1}^{n} (\dot{\boldsymbol{r}}'_i \times m_i(\boldsymbol{v}_G + \dot{\boldsymbol{r}}'_i)) + \sum_{i=1}^{n} (\boldsymbol{r}'_i \times m_i \dot{\boldsymbol{v}'}_i) = \boldsymbol{0} + \sum_{i=1}^{n} (\boldsymbol{r}'_i \times \boldsymbol{F}_i) + \sum_{i=1}^{n} M_i \ . \tag{15}$$

In the above equation, the term $\dot{\boldsymbol{r}}'_i \times m_i \dot{\boldsymbol{r}}'_i$ is clearly zero, and $\sum_{i=1}^{n} (\dot{\boldsymbol{r}}'_i \times m_i \boldsymbol{v}_G) = -\boldsymbol{v}_G \times \sum_{i=1}^{n} m_i \dot{\boldsymbol{r}}'_i = -\boldsymbol{v}_G \times d(\sum_{i=1}^{n} m_i \boldsymbol{r}'_i)/dt = \boldsymbol{0}$. Thus, we have that

$$\dot{\boldsymbol{H}}_G = \sum_{i=1}^{n} (\boldsymbol{r}'_i \times m_i \dot{\boldsymbol{v}'}_i) = \boldsymbol{M}_G \ . \tag{16}$$

Here, $\boldsymbol{M}_G = \sum_{i=1}^{n}(\boldsymbol{r}'_i \times \boldsymbol{F}_i) + \sum_{i=1}^{n} M_i$, is the total moment, about G, of the applied external forces plus any external moments. Note that external forces in general produce unequal moments about O and G while applied external moments (torques) produce the same moment about O and G.

The above expression is very powerful and allows us to solve, with great simplicity, a large class of problems in rigid body dynamics. Its power lies in the fact that it is applicable in very general situations: In the derivation of equation (16), we have made no assumptions about the motion of the center of mass, G. That is, equation (16) is valid even when G is *accelerated*. We have implicitly assumed that the reference frame used to describe $\boldsymbol{r}'_i$ in equation 13 is non-rotating with respect to the fixed frame xyz (otherwise, we would have written $\dot{\boldsymbol{r}}'_i = \boldsymbol{v}'_i + \omega' \times \boldsymbol{r}'_i$, with ω', the angular velocity of the frame considered). It is not difficult to show that equation (16) is still valid if the reference frame rotates, provided the angular velocity is *constant*. If the reference frame rotates with a constant angular velocity, the angular momentum will differ from that of equations (12) and (14) by a constant, but equation 16 still will be valid.

Finally, by combining equations 30 and 12, the angular momentum about a fixed point, O, can be expressed as a function of the angular momentum about the center of mass, as,

$$\boldsymbol{H}_O = \boldsymbol{r}_G \times m\boldsymbol{v}_G + \boldsymbol{H}_G. \tag{17}$$

Just as we could incorporate collisions in our statement of conservation of linear momentum, we can incorporate collision in our statement of conservation of angular momentum. Collisions conserve both linear and angular momentum. Just as changes in linear momentum result for linear impulses, changes in angular momentum result from angular impulses.

Rocket

A *rocket* is a vehicle that propels itself through space by ejecting a propellant gas at high speed in a direction opposite the desired direction of motion. The German V-2 rocket was an early example, as were the United States rockets such as Juno, Redstone, Agena, and Saturn. The largest and most powerful rocket ever used is the United States Saturn V Moon rocket, which took the Apollo astronauts to the Moon in the 1960s and 1970s.

In order to place a spacecraft into low-Earth orbit, a rocket must accelerate its payload from rest to a speed of about 17,000 miles per hour. In order to reach this speed, most of the rocket's mass must be fuel. The amount of fuel required for a given mass of payload is governed by the *rocket equation*, which will be derived here.

Some critics of early space exploration claimed that rockets would not be able to travel in space because "they would have nothing to push against." As we'll see here, such arguments are silly—one needs only to make use of the conservation of momentum to show that rockets can work in space.

Derivation of the Equation

Let's now derive the rocket equation. Given a rocket of mass m, we will wish to find an equation that tells us how much fuel (propellant) is required to change the rocket's speed by an amount Δv. The complication here is that the rocket loses mass as it expels propellant, so we need to allow for that.

Suppose that at an initial time $t = 0$, a rocket has velocity v and total mass m, including propellant mass. The total momentum of the rocket and propellant at time $t = 0$ is therefore mv

expulsion of propellant will cause the rocket to then have mass $m + dm$ and velocity $v + dv$. The total momentum of the system at $t = dt$ is then the sum of the rocket and propellant momenta, $(m + dm)(v + dv) + (v - v_p)(-dm)$. By conservation of momentum, the momentum of the system at time $t = 0$ must equal the momentum at time $t = dt$:

$$
\begin{aligned}
mv &= (m + dm)(v + dv) + (v - v_p)(-dm) && (1)\\
&= mv + v\,dm + m\,dv + dm\,dv - v\,dm + v_p\,dm && (2)\\
& && (3)
\end{aligned}
$$

Now the two mv terms cancel, the two $v\,dm$ terms cancel, and the term $dm\,dv$ is a second-order differential, which can also be cancelled. We're then left with

$$
\begin{aligned}
0 &= m\,dv + v_p\,dm && (4)\\
m\,dv &= -v_p\,dm && (5)\\
dv &= -v_p\,\frac{dm}{m} && (6)
\end{aligned}
$$

Now let the rocket burn all its propellant. The rocket's velocity will change by a total amount Δv and its mass will change from m to its empty mass m_e. Integrating Eq. (6) over the entire propellant burn, we find

$$\int_v^{v+\Delta v} dv = -v_p \int_m^{m_e} \frac{dm}{m} \tag{7}$$

Or, evaluating the integrals,

$$\Delta v = -v_p \ln \frac{m_e}{m} \tag{8}$$

or

$$\boxed{\Delta v = v_p \ln \frac{m}{m_e}} \tag{9}$$

Eq. (9) is called the *rocket equation*. It relates the fueled and empty masses of the rocket and the velocity of the propellant to the total change in velocity of the rocket.

Example

Let's take as an example the launch of a rocket from the Earth's surface to low-Earth orbit. In this case, the rocket's velocity will need to change by an amount Δv = 17,000 mph, or about 7600 m/s. Let's say we have a rocket that can expel propellant with a speed $v_p = 4000$ m/s. Then by Eq. (11),

$$1 - \frac{m_e}{m} = 1 - e^{-\Delta v / v_p} = 0.85, \tag{12}$$

so 85% of the rocket's initial mass must be propellant.

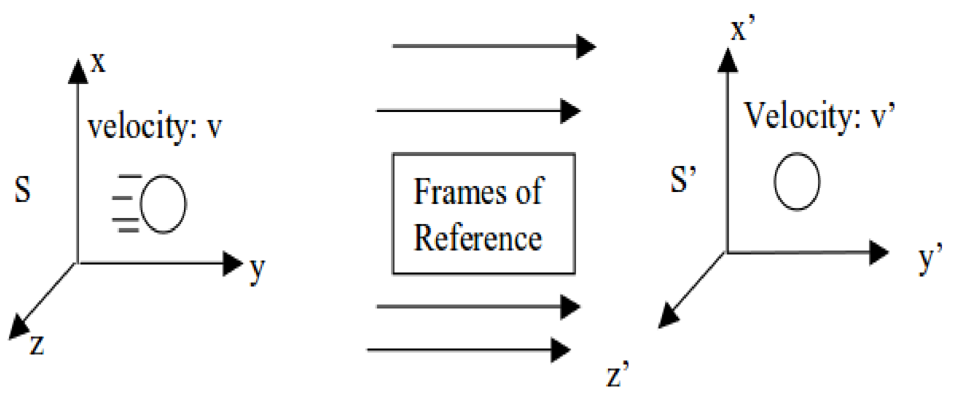
Staging

In practice, it is found that it can be more efficient to launch rockets in *stages*, where part of the rocket structure drops away when it is no longer needed, thus decreasing the amount of mass that needs to be placed in orbit. For example, the Saturn V rocket had three stages. The large lower first stage contained a large fuel tank and large engines. When all the fuel contained in that stage had been spent, the entire first stage separated and dropped away, and a smaller second stage was ignited. When all the second-stage fuel was spent, it too separated and and dropped away, and the third stage engine ignited, which placed the spacecraft into Earth orbit. This staged approach requires much less fuel than launching the entire Saturn V rocket into orbit.

Reference Frames

To describe a physical event, we need to establish a 3-dimensional coordinate system associated with measurement.

Let's consider two reference frames. One, called S, is shown at left. The other, called S', is shown at right. Let's imagine a meatball which is moving with velocity 'v' within reference frame S. Reference frame S' may be moving with respect to frame S, so the perceived velocity of the meatball in S' may be different than an observer in S would measure. In fact, v' could be zero. So the value of the y' coordinate may also differ in the two reference frames. However, as drawn, the x (x') and z (z') coordinates would be the same in the two reference frames.



Inertial Reference Frames

Inertia: An object moves at constant velocity unless acted upon by an external force.

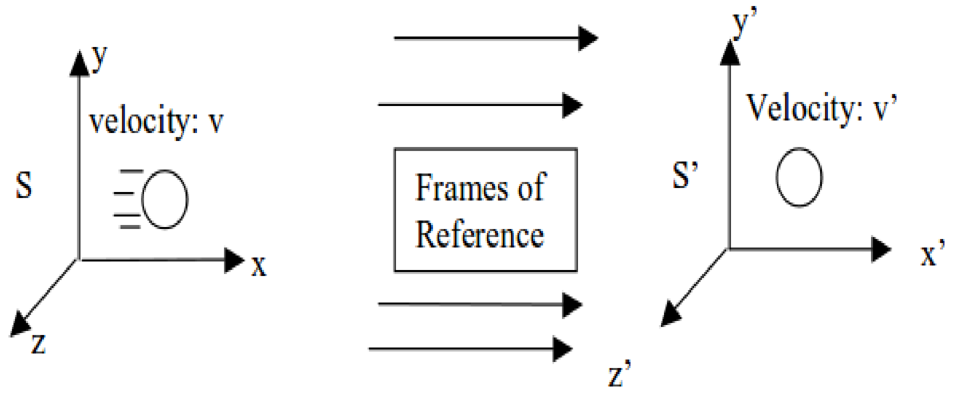
Given the concept of inertia, we find it useful to talk about 'inertial reference frames' which are three-dimensional coordinate systems which travel at constant velocity. In such a frame, an object is observed to have no acceleration when no forces are acting on it. If a reference frame moves with constant velocity relative to an inertial reference frame, it also is an inertial reference frame. There is no absolute inertial reference frame, meaning that there is no state of velocity which is special in the universe. All inertial reference frames are equivalent. One can only detect the relative motion of one inertial reference frame to another.

Principle of Galilean Relativity

"Laws of mechanics must be same in all inertial frames of reference"

Galilean Transformations

Consider a meatball in frame S moving with velocity, v, within that frame, and S' is moving with velocity V' relative to frame S. This is shown in the following Figure.



We want to know how to determine the coordinates in S' when we know them in frame S. In the picture above, in the S frame the meatball is moving and the {x,y,z} axes are fixed. When we transform to the S' coordinate system (so that {x',y',z'} are at rest), it now looks like the meatball has velocity v' & that the old axes {x,y,z} are moving with velocity v in the NEGATIVE x' direction. "v" then appears as the relative velocity of the PRIMED coordinate system {x',y',z'}, S', compared to the UNPRIMED coordinate system {x,y,z} or S.

To determine the coordinates of the meatball in one frame, S', when we know it's coordinates in another frame, S, we employ the Galilean space and time transformations.

Galilean space-time transformations:

If S' has a velocity relative to S so that v' = 0, then we have

$x' = x+vt$
$y'=y$
$z'=z$
$t'=t$

(Note: remember, v or v' are vectors, so they have a sign.)
The time interval between any two events is the same in any frame of reference.

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Thanks

UNIT II

Inverse Force fields

Inverse-square law is a physical law implying that a specified physical quantity or intensity shows inverse proportionality to the square of the distance from the source of that physical quantity. The fundamental cause for this can be understood as geometric dilution corresponding to point-source radiation into three-dimensional space. This comes from strictly geometrical considerations. The intensity of the influence at any given radius r is the source strength divided by the area of the sphere. Being strictly geometric in its origin, the inverse square law applies to diverse phenomena. Point sources of gravitational force, electric field, light, sound or radiation obey the inverse square law. It is a subject of continuing debate with a source such as a skunk on top of a flag pole; will it's smell drop off according to the inverse square law? Mathematically formulated:

$$\boxed{\textbf{Intensity} \propto \frac{1}{\textbf{distance}^2}}$$

It can further be extended as

$$\boxed{\frac{\textbf{Intensity}_1}{\textbf{Intensity}_2} = \frac{{\textbf{distance}_2}^2}{{\textbf{distance}_1}^2}}$$

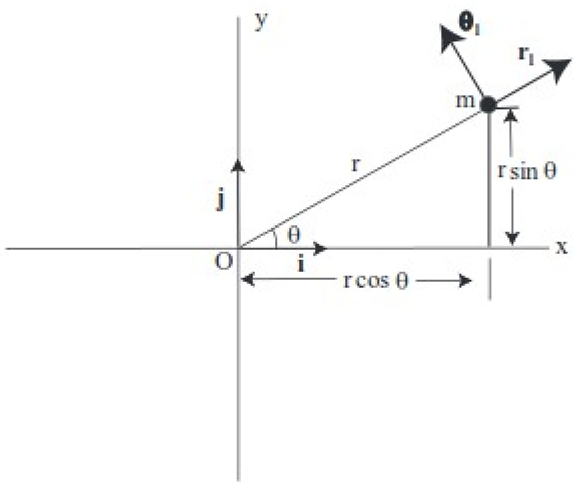
Or,

$$\boxed{\text{Intensity}_1 * {\text{distance}_1}^2 = \text{Intensity}_2 * {\text{distance}_2}^2}$$

The divergence of a field(vector field) which is the resultant of radial inverse-square law fields with respect to definite sources is everywhere proportional to the strength of the local sources, and hence zero outside sources. Newtons gravitational law follows an inverse-square law, like the effects of electric, light, magnetic, sound, and radiation phenomena.

Equation of Orbit

the motion of the particle must occur in a plane, which we take as the xy plane, and the center of force is taken as the origin. In Fig. we show the xy plane, as well as the polar coordinate system in the plane.



Fig(1):Polar coordinate system of a particle moving in the xy plane.
Since the vectorial nature of the central force is expressed in terms of a radial vector from the origin it is most natural (though not required!) to write the equations of motion in polar coordinates. In earlier lectures we derived the expression for the acceleration of a particle in polar coordinates

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{r}_1 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{\theta}_1. \qquad (1)$$

Then, using Newton's second law, and the mathematical form f or the central force given as,

$$\mathbf{F} = f(r)\mathbf{r}_1 = f(r)\frac{\mathbf{r}}{r}, \qquad (2)$$

we have:

$$m(\ddot{r} - r\dot{\theta}^2)\mathbf{r}_1 + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{\theta}_1 = f(r)\mathbf{r}_1, \tag{3}$$

$$m(\ddot{r} - r\dot{\theta}^2) = f(r), \tag{4}$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0. \tag{5}$$

These are the basic equations of motion for a particle in a central force field. From these we get,

$$r^2\dot{\theta} = h = \text{constant}. \tag{6}$$

This constant of the motion will allow you to determine the θ component of motion, provided you know the r component of motion. However, (4) and (5) are coupled (nonlinear) equations for the r and θ components of the motion. How could you solve them without solving for both the r and θ components? This is where alternative forms of the equations of motion are useful. Let us rewrite (8) in the following form (by dividing through by the mass m):

Let us rewrite (4) in the following form (by dividing through by the mass m):

$$\ddot{r} - r\dot{\theta}^2 = \frac{f(r)}{m}. \tag{7}$$

Now, using (6), (7) can be written entirely in terms of r:

$$\ddot{r} - \frac{h^2}{r^3} = \frac{f(r)}{m}. \tag{8}$$

We can use (8) to solve for r(t), and the use (6) to solve for θ(t). Equation (8) is a nonlinear equation. There is a useful change of variables, which for certain important central forces, turns the equation into a linear differential equation with constant coefficients, and these can always be solved analytically. Here we describe this coordinate transformation.

Let

$$r = \frac{1}{u}.$$

This is part of the coordinate transformation. We will also use
θ as a new "time" variable. Coordinate transformation are effected by the chain rule, since this allows us to express derivatives of "old" coordinates in terms of the "new" coordinates. We have:

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{h}{r^2}\frac{dr}{d\theta} = \frac{h}{r^2}\frac{dr}{du}\frac{du}{d\theta} = -h\frac{du}{d\theta}, \qquad (9)$$

And

$$\ddot{r} = \frac{d\dot{r}}{dt} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right) = \frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right)\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2}, \qquad (10)$$

where, in both expressions, we have used the relation
$r^2\dot{\theta}=h$ at strategic points.

Now

$$r\dot{\theta}^2 = r\frac{h^2}{r^4} = h^2u^3. \qquad (11)$$

Substituting this relation, along with (10) into (4), gives

$$m\left(-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3\right) = f\left(\frac{1}{u}\right),$$

or

$$\frac{d^2u}{d\theta^2} + u = -\frac{f\left(\frac{1}{u}\right)}{mh^2u^2}. \qquad (12)$$

$$f(r) = \frac{K}{r^2},$$

Now if

where K is some constant (12)becomes a linear, constant coefficient equation

Exact solution for the orbit equation for an inverse square law force

Consider now the orbit equation for u(θ) for the attractive inverse square law force, $F_r(r)=-k/r^2$, namely

$$u''(\theta) + u = \frac{k}{mh^2}. \qquad (13)$$

In Newtonian gravity k=GMm so the r.h.s becomes GM/h2.

We recognise that above equation is just the equation of a simple harmonic oscillator with frequency ω= 1. The general solution is:

$$u(\theta) = A\cos(\theta) + B\sin(\theta) + \frac{k}{mh^2} \qquad (14)$$

Let us now determine A and B for specific initial conditions (yet the conclusions we shall draw concerning the possible orbit solutions remain general). Consider a case where a particle of mass m is projected at distance a from the centre of the force with velocity v, in a direction perpendicular to the radius vector from the centre to the projection point. Without loss of generality we define θ = 0 to correspond to time t= 0. We then have, at t= 0, tangent velocity $v_\theta = a\dot{\theta} = v$ while the radial velocity vanishes, $v_r = 0$. The tangent motion simply fixes the angular momentum: according to (9) $h = r^2\dot{\theta} = av_\theta = av$. Using the initial condition for the radial velocity together with (9) we have:

$$u'(\theta = 0) = -\frac{\dot{r}}{h} = -\frac{v_r}{h} = 0 \qquad (15)$$

This should be compared to the derivative of u according to our solution (14)

$$u'(\theta) = -A\sin(\theta) + B\cos(\theta) \tag{16}$$

leading to the conclusion that B= 0. Finally the initial condition for the position yields;

$$u(\theta) = A\cos(0) + B\sin(0) + \frac{k}{mh^2} = \frac{1}{a}$$

$$A = \frac{1}{a} - \frac{k}{mh^2} \tag{17}$$

We therefore get the following orbit solution

$$\frac{1}{r} = \left(\frac{1}{a} - \frac{k}{mh^2}\right)\cos(\theta) + \frac{k}{mh^2}$$

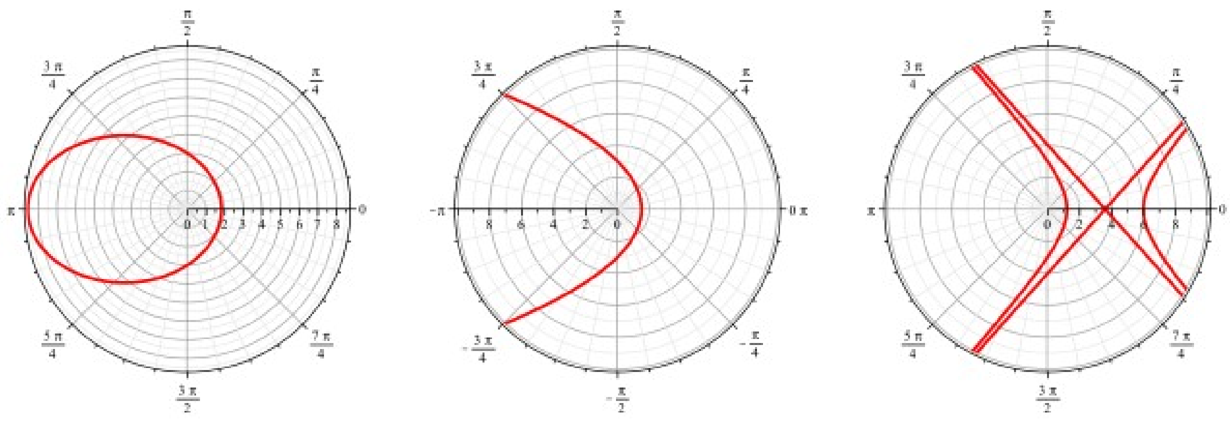
Or

$$r = \frac{l}{1 + \epsilon\cos(\theta)} \tag{18}$$

where we defined

$$l \equiv \frac{mh^2}{k}, \qquad \epsilon \equiv \frac{l}{a} - 1 \tag{19}$$

We recognise that our solution for the orbit (18) describes a conic section where r measures the distance from the focal point to the orbit (the centre of the force is at the focal point). Depending on the value of the eccentricity(a dimensionless parameter), it is a circle (€= 0), an ellipse (0<€<1), a parabola (€= 1), or a hyperbola (€>1).



Fig(2): The three conic sections described by eq. (18), here shown for l=

3 and €=0.61 (ellipse), €=1(parabola) and €=1.5 (hyperbola) . Note that in all cases the origin (r= 0) corresponds to the centre of force: this is a focal point of these conic section

- a circular orbit $r = l$ for $\epsilon = 0$, that is for $a = \frac{k}{mv_\theta^2} = \frac{GM}{v_\theta^2}$.
- an elliptic orbit $(0 < \epsilon < 1)$ for $\frac{GM}{v_\theta^2} < a < \frac{2GM}{v_\theta^2}$.
- a parabolic orbit $(\epsilon = 1)$ for $a = \frac{2GM}{v_\theta^2}$.
- a hyperbolic orbit $(\epsilon > 1)$ for $a > \frac{2GM}{v_\theta^2}$

Kepler's Laws

The motions of the planets, as they seemingly wander against the background of the stars, have been a puzzle since the dawn of history. The "loop-the-loop" motion of Mars, shown in Fig(3). below, was particularly baffling. Johannes Kepler (I57I-1630), after a lifetime of study,worked out the empirical laws that govern these motions. Tycho Brahe (1546- 1601), the last of the great astronomers to make observations without the help of a telescope, compiled the extensive data from which Kepler was able to derive the three laws of planetary motion that now bear Kepler's name. Later, Newton (1642-1727) showed that his law of gravitation leads to Kepler's laws.

In this section we discuss each of Kepler's three laws. Although here we apply the laws to planets orbiting the Sun, they hold equally well for satellites, either natural or artiflcial, orbiting Earth or any other massive central body.

1. THE LAW OF ORBITS: All planets move in elliptical orbits. with the Sun at one focus. Figure (3) shows a planet of mass m moving in such an orbit around the Sun, whose mass is M.We assume that M * m, so that the center of mass of the planet - Sun system is approximately at the center of the Sun. The orbit in Fig. (3) is described by giving its semi-major axis a and its eccentricity e, the latter defined so that ea ts the distance from the center of the ellipse to either focus F or F'. An eccentricity of zero corresponds to a circle, in which the two foci merge to a single central point. The eccentricities of the planetary orbits are not large; so if the orbits are drawn to scale, they look circular. The eccentricity of the ellipse of Fig. (3), which has been exaggerated for clarity, is 0.74.The eccentricity of Earth's orbit is only 0.0167 .

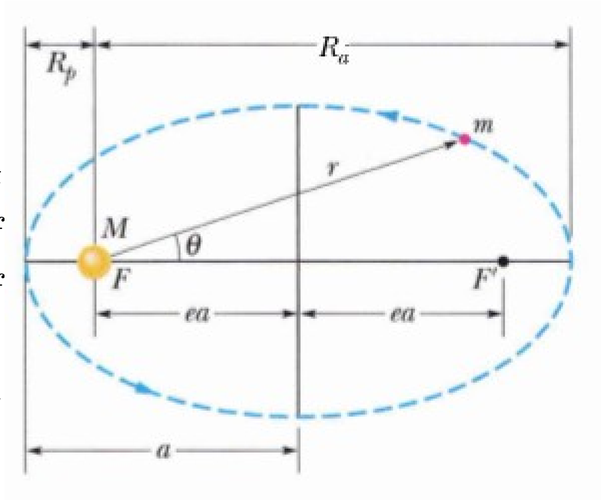
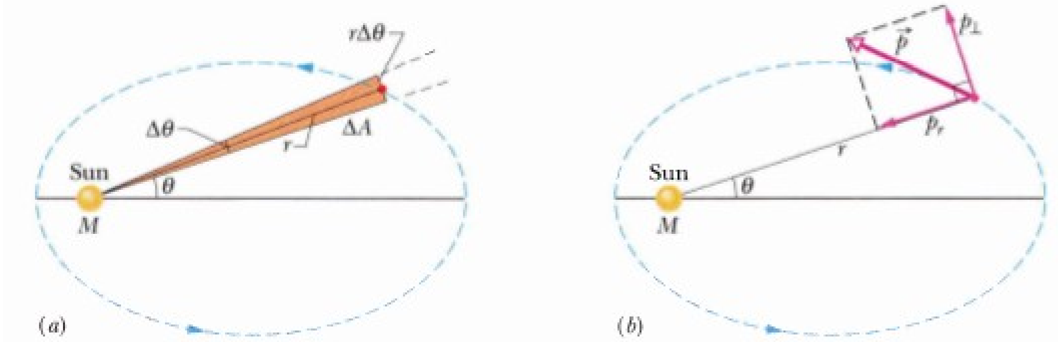


Fig (3):A planet of mass m moving in an elliptical orbit around the Sun. The Sun, of mass M,is at one focus F of the ellipse. The other focus is F', which is located in empty space. Each focus is a distance ea from the ellipse's center, with e being the eccentricity of the ellipse The semimajor axis a of. the ellipse, the perihelion (nearest the Sun) distance R_p*, and the aphelion (farthest from the Sun) distance* R_a *are also shown.*

THE LAW OF AREAS: A line that connects a planet to the Sun sweeps out equal areas in the plane of the planet's orbit in equal time intervals; that is, the rate dA/dt at which it sweeps out area A is constant. Qualitatively, this second law tells us that the planet will move most slowly when it is farthest from the Sun and most rapidly when it is nearest to the Sun. As it turns out, Kepler's second law is totally equivalent to the law of conservation of angular momentum. Let us prove it.



Fig(4) (a) In time Δt*,the line r connecting the planet to the Sun moves through an angle* $\Delta\theta$*, sweeping out an area* ΔA *(shaded) .(b) The linear* *momentum $\vec{p}$ of the planet and the components of $\vec{p}$.*

The area of the shaded wedge in Fig. (4)a closely approximates the area swept out in time Δt by a line connecting the Sun and the planet, which are separated by distance r. The area ΔA of the wedge is approximately the area of a triangle with base rΔθ and height r. Since the area of a triangle is one-half of the base times the height, $\Delta A = \frac{1}{2} r^2 \theta$. This expression for ΔA becomes more exact as Δt (hence Δθ) approaches zero. The instantaneous rate at which area is being swept out is then

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \omega, \tag{1}$$

in which ω is the angular speed of the rotating line connecting Sun and planet. Figure(4)b shows the linear momentum $\vec{p}$ the planet, along with the radial and perpendicular components of i. Also we know ($L = rp_\perp$), the magnitude of the angular momentum $\vec{L}$ of the planet about the Sun is given by the product of r and $P_\perp$,the component of $\vec{p}$ perpendicular to r. Here, for a planet of mass m,

$$\begin{aligned} L &= rp_\perp = (r)(mv_\perp) = (r)(m\omega r) \\ &= mr^2\omega, \end{aligned} \tag{2}$$

Where v = rω

Eliminating $r^2\omega$ between Eqs.(1) and (2) leads to

$$\frac{dA}{dt} = \frac{L}{2m}. \tag{3}$$

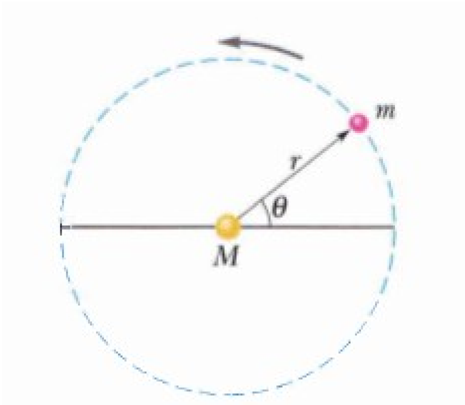
It dA/dt rs constant, as Kepler said it is, then Eq.(3) means that L must also be constant-angular momentum is conserved. Kepler's second law is indeed equivalent to the law of conservation of angular momentum.

3. THE LAW O F PERIODS

The square of the periods is proportional to the c ube of the semimajor axis of its orbit.

To see this, consider the circular orbit of Fig. (5), with radius r (the

radius of
a circle is equivalent to the semimajor axis of an ellipse).



Fig(5)A planet of mass m moving around the Sun in a circular orbit of radius r.

Applying Newton's second law (F = ma) to the orbiting planet in Fig. (5) yields

$$\frac{GMm}{r^2} = (m)(\omega^2 r).$$

Here a = $r^2\omega$ = $2\pi/T$ is known as the centripetal acceleration. T represents the period of motion. Thus we obtain Kepler's third law as;

$$T^2 = \left(\frac{4\pi^2}{GM}\right)r^3$$

This is known as the law of periods. The quantity in parentheses is a constant that depends only on the mass M of. The central body about which the planet orbits. Above Equation holds also for elliptical orbits, provided we replace r with a, the semimajor axis of the ellipse.

Gravitational law and Field

Physicists like to study seemingly unrelated phenomena to show that a relationship can be found if the phenomena are examined closely enough. This search for unification has been going on for centuries. In 1665, the 23-year-old Isaac Newton made a basic contribution to physics when he showed that the force that holds the Moon in its orbit is the same force that makes an apple fall. We take this knowledge so much for granted now that it is not easy for us to comprehend the ancient belief that the motions of earthbound bodies and heavenly bodies were different in kind and were governed by different laws. Newton concluded not only that Earth attracts both apples and the Moon but also that every body in the universe attracts every other body; this tendency of bodies to move toward each other is called gravitation. Newton's conclusion takes a little getting used to, because the familiar attraction of Earth for earth-bound bodies is so great that it overwhelms the attraction that earthbound bodies have for each other. For example, Earth attracts an apple with a force magnitude of about 0.8 N. You also attract a nearby apple (and it attracts you), but the force of attraction has less magnitude than the weight of a speck of dust. Newton proposed a force law that we call Newton's law of gravitation: Every particle attracts any other particle with a gravitational force of magnitude

$$F = G\frac{m_1 m_2}{r^2}$$

Here m_1 and m_2 are the masses of the particles, r is the distance between them, and G is the gravitational constant, with a value that is now known to be

$$G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2} \\ = 6.67 \times 10^{-11}\ \mathrm{m^3/kg \cdot s^2}.$$

In vector form we can write;

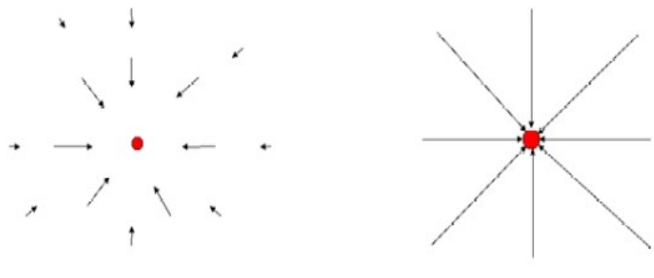
$$\vec{F} = G\frac{m_1 m_2}{r^2}\hat{\mathrm{r}}.$$

The strength of the gravitational force that is, how strongly two particles with given masses at a given separation attract
each other depends on the value of the gravitational constant G. If G by some miracle-were suddenly multiplied by a factor of 10, you would be crushed to the floor by Earth's attraction. If G were divided by this

factor, Earth's attraction would be so weak that you could jump over a building. Although Newton's law of gravitation applies strictly to particles, we can also apply it to real objects as long as the sizes of the objects are small relative to the distance between them. The Moon and Earth are far enough apart so that, to a good approximation, we can treat them both as particles-but what about an apple and Earth? From the point of view of the apple, the broad and level Earth, stretching out to the horizon beneath the apple, certainly does not look like a particle.

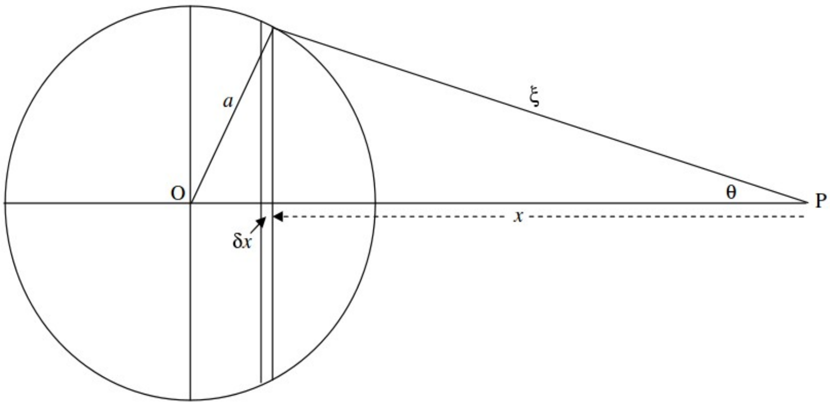
A **gravitational field** is the force field that exists in the space around every mass or group of masses. This field extends out in all directions, but the magnitude of the gravitational force decreases as the distance from the object increases. It is measured in units of force per mass, usually newtons per kilogram (N/kg). A gravitational field is a type of force field and is analogous to electric and magnetic fields for electrically charged particles and magnets, respectively.

There are two ways of showing the gravitational field around an object: with arrows and with field lines. Both of these are shown in the picture below. Arrows show the magnitude and direction of the force at different points in space. The longer the arrow, the greater the magnitude. Field lines show the direction the force would act on an object placed at that point in space. The magnitude of the field is represented by the spacing of the lines. The closer the lines are to each other, the higher the magnitude.



The gravitational field varies slightly at the Earth's surface. For example, the field is slightly stronger than average over subterranean lead deposits.

Field due to spherical shell



We imagine a hollow spherical shell of radius a, surface density σ, and a point P at a distance r from the centre of the sphere. Consider an elemental zone of thickness δx. The mass of this element is 2πaσ δx. (In case you doubt this, or you didn't know, "the area of a zone on the surface of a sphere is equal to the corresponding area projected on to the circumscribing cylinder".) The field due to this zone, in the direction PO is

$$\frac{2\pi a\sigma G\cos\theta\,\delta x}{\xi^2}.$$

Let's express this all in terms of a single variable, ξ. We are going to have to express x and θ in terms of ξ. We have ,

$$a^2 = r^2 + \xi^2 - 2r\xi\cos\theta = r^2 + \xi^2 - 2rx,$$

from which,

$$\cos\theta = \frac{r^2 - a^2 + \xi^2}{2r\xi} \qquad \text{and} \qquad \delta x = \frac{\xi\,\delta\xi}{r}.$$

Therefore the field at P due to the zone is $\frac{\pi a G\sigma}{r^2}\left(1 + \frac{r^2 - a^2}{\xi^2}\right)\delta\xi..$

If P is an external point, in order to find the field due to the entire spherical shell, we integrate from ξ = r—a to r+ a This results in

$$g = \frac{GM}{r^2}.$$

But if P is an internal point, in order to find the field due to the entire spherical shell, we integrate from ξ = a —r to a+ r, which results in g=0. Thus we have the important result that the field at an external point due to a hollow spherical shell is exactly the same as if all the mass were concentrated at a point at the centre of the sphere, whereas the field inside the sphere is zero.

Potential due to spherical shell

Outside the sphere, the field and the potential are just as if all the mass were concentrated at a point in the centre. The potential, then, out side the sphere, is just —GM/r.

Inside the sphere, the field is zero and therefore the potential is uniform and is equal to the potential at the surface, which is —GM/a. The reader should draw a graph of the potential as a function of distance from centre of the sphere. There is a discontinuity in the slope of the potential (and hence in the field) at the surface.

Field due to solid Sphere

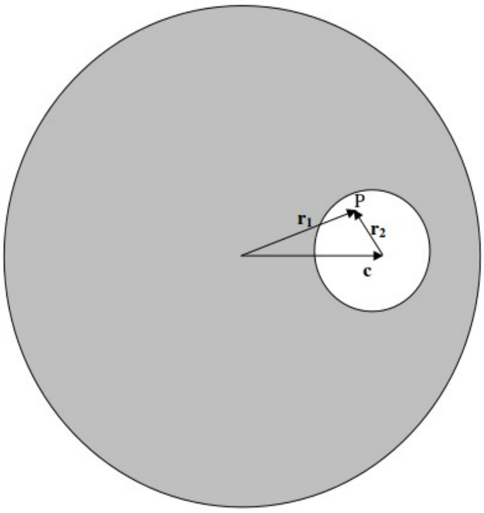
A solid sphere is just lots of hollow spheres nested together. Therefore, the field at an external point is just the same as if all the mass were concentrated at the centre, and the field at an internal point P is the same is if all the mass interior to P, namely M_r, were concentrated at the centre, the mass exterior to P not contributing at all to the field at P. This is true not only for a sphere of uniform density, but of any sphere in which the density depends only of the distance from the centre – i.e., any spherically symmetric distribution of matter.

If the sphere is uniform, we have $\frac{M_r}{M} = \frac{r^3}{a^3}$, so the field inside is

$$g = \frac{GM_r}{r^2} = \frac{GMr}{a^3}$$

Thus, inside a uniform solid sphere, the field increases linearly from zero at the centre to GM/a^2 at the surface, and thereafter it falls off as GM/r^2. If a uniform hollow sphere has a narrow hole bored through it, and a small particle of mass m is allowed to drop through the hole, the particle will experience a force towards the centre of $GMmr/a^3$, and will consequently oscillate with period P given by

$$p^2 = \frac{4\pi^2}{GM}a^3$$



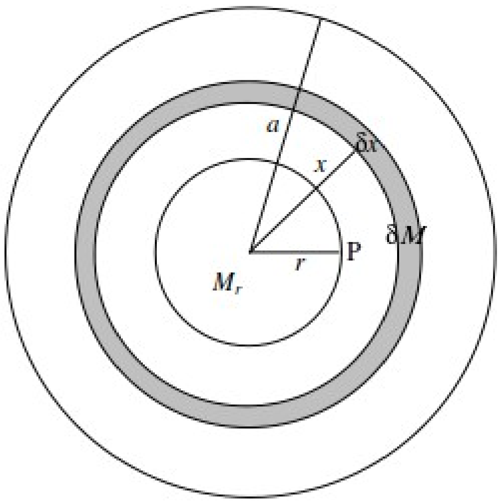
P is a point inside the bubble. The field at P is equal to the field due to the entire sphere minus the field due to the missing mass of the bubble. That is, it is

$$\mathbf{g} = -\tfrac{4}{3}\pi G\rho\mathbf{r}_1 - (-\tfrac{4}{3}\pi G\rho\mathbf{r}_2) = -\tfrac{4}{3}\pi G\rho(\mathbf{r}_1 - \mathbf{r}_2) = -\tfrac{4}{3}\pi G\rho\mathbf{c}.$$

That is, the field at P is uniform (i.e. is independent of the position of P) and is parallel to the line joining the centres of the two spheres.

Potential due to solid Sphere

The potential outside a solid sphere is just the same as if all the mass were concentrated at a point in the centre. This is so, even if the density is not uniform, and long as it is spherically distributed. We are going to find the potential at a point P inside a uniform sphere of radius a, mass M, density ρ,



at a distance r from the centre (r< a). We can do this in two parts. First, there is the potential from that part of the sphere “below” P. This is -Gm_r/r Where $M_r = \frac{r^3 M}{a^3}$ is the mass within radius r. Now we need to deal with the material “above” P. Consider a spherical shell of radii x, x+ δx. Its mass is

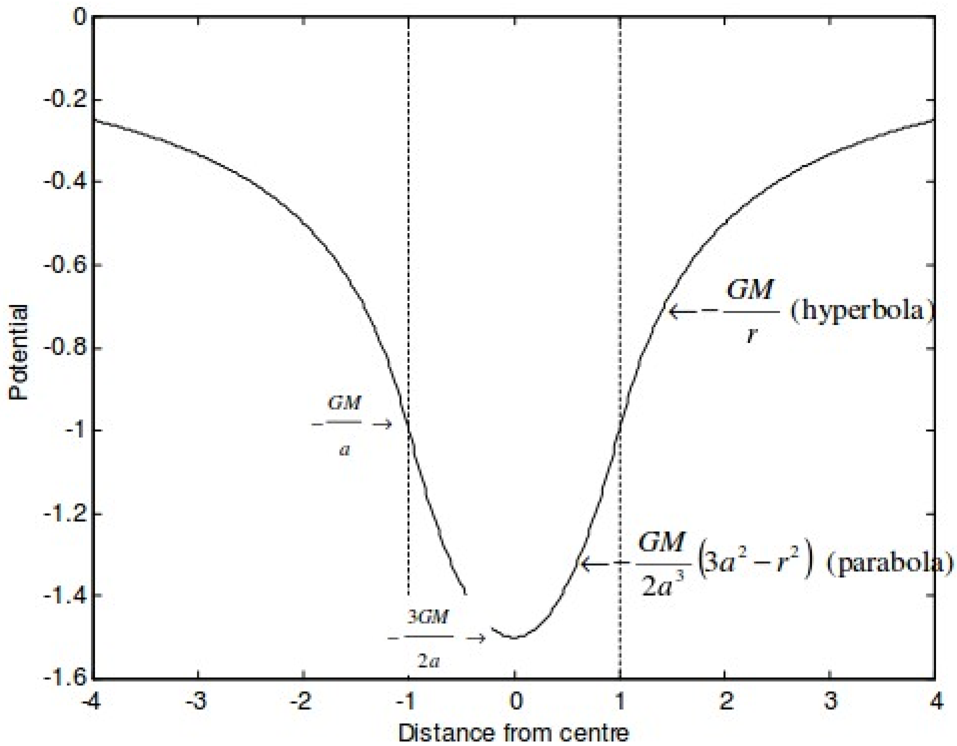
$$\delta M = \frac{4\pi x^2 \delta x}{\frac{4}{3}\pi a^3} \cdot M = \frac{3Mx^2 \delta x}{a^3}.$$

The potential from this shell is

$$-G\delta M/x = -\frac{3GMx\delta x}{a^3}.$$

This is to be integrated from x= 0 to a, and we must then add the contribution from the material “below” P . The final result is

$$\psi = -\frac{GM}{2a^3}\left(3a^2 - r^2\right).$$



Fig(5) shows the potential both inside and outside a uniform solid *sphere. The potential is in units of $-GM/r$, and distance is in units of a, the radius of the sphere.*

Rigid Body motion

A rigid body is a collection of particles with fixed relative positions, independent of the motion carried out by the body. The dynamics of a rigid body has been discussed in our introductory courses, and the techniques discussed in these courses allow us to solve many problems in which the motion can be reduced to two-dimensional motion. In this special case, we found that the angular momentum associated with the rotation of the rigid object is directed in the same direction as the angular velocity:

$$\bar{L} = I\bar{\omega}$$

In this equation, I is the moment of inertia of the rigid body which was defined as

$$I = \sum_i m_i r_i^2$$

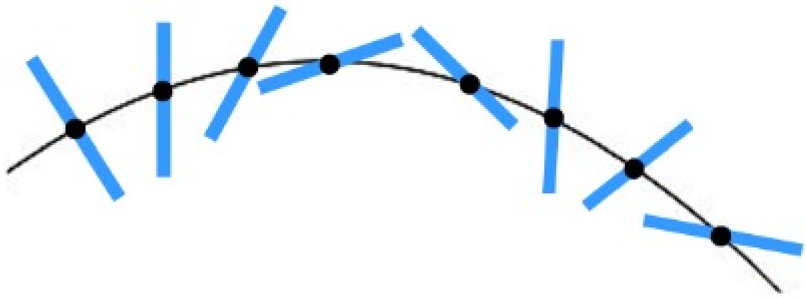
where r_i is the distance of mass m_i from the rotation axis. We also found that the kinetic energy of the body, associated with its rotation, is equal to

$$T = \frac{1}{2} I \omega^2$$

The complexity of the motion increases when we need three dimensions to describe the motion.

In classical mechanics a rigid body is usually considered as a continuous mass distribution, while in quantum mechanics a rigid body is usually thought of as a collection of point masses. For instance, in quantum mechanics molecules (consisting of the point masses: electrons and nuclei) are often seen as rigid bodies

The general motion of a rigid body of mass m consists of a translation of the center of mass with velocity V_{cm} and a rotation about the center of mass with all elements of the rigid body rotating with the same angular velocity ω_{cm}. Figure(6) shows the center of mass of a thrown rigid rod follows a parabolic trajectory while the rod rotates about the center of mass.



Fig(6)The center of mass of a thrown rigid rod follows a parabolic trajectory while the rod rotates about the center of mass.

Rigid body rotation

Consider a rigid body executing pure rotational motion (*i.e.*, rotational motion which has no translational component). It is possible to define an *axis of rotation* (which, for the sake of simplicity, is assumed to pass through the body)--this axis corresponds to the straight-line which is the locus of all points inside the body which remain stationary as the body rotates. A general point located inside the body executes *circular motion* which is centred on the rotation axis, and orientated in the plane perpendicular to this axis. In the following, we tacitly assume that the axis of rotation remains fixed.

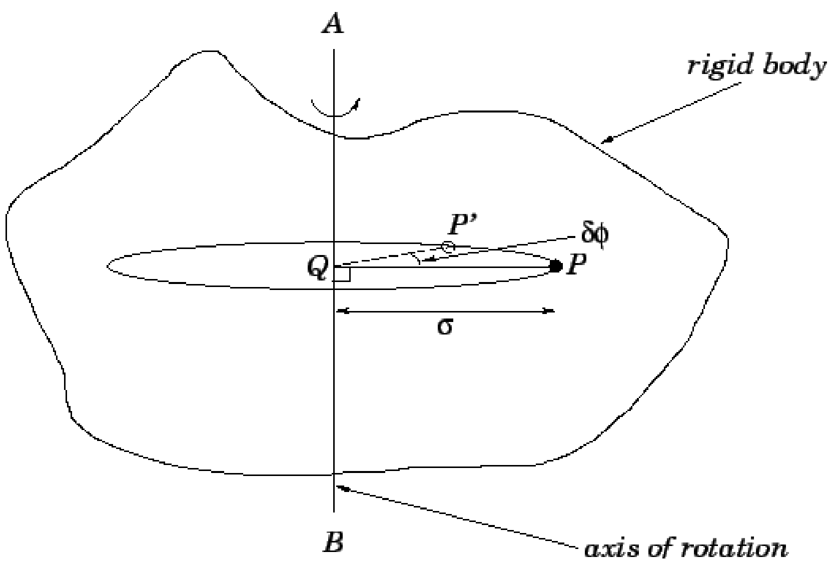


Figure above shows a typical rigidly rotating body. The axis of rotation is the line AB. A general point P lying within the body executes a circular orbit, centred on AB, in the plane perpendicular to AB. Let the line QP be a radius of this orbit which links the axis of rotation to the instantaneous position of P at time t. Obviously, this implies that QP is normal to AB. Suppose that at time $t + \delta t$ point P has moved to P', and the radius QP has rotated through an angle $\delta\phi$. The instantaneous *angular velocity* of the body $\omega(t)$ is defined

$$v = \sigma\,\omega,$$

where σ is the *perpendicular distance* from the axis of rotation to point P. Thus, in a rigidly rotating body, the rotation speed increases *linearly* with (perpendicular) distance from the axis of rotation.
It is helpful to introduce the *angular acceleration* $\alpha(t)$ of a rigidly rotating body: this quantity is defined as the time derivative of the angular velocity. Thus,

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\phi}{dt^2},$$

where ϕ is the angular coordinate of some arbitrarily chosen point reference within the body, measured with respect to the rotation axis.

Note that angular velocities are conventionally measured in radians per second, whereas angular accelerations are measured in radians per second squared.
For a body rotating with constant angular velocity, ω, the angular acceleration is zero, and the rotation angle ϕ increases linearly with time:

$$\phi(t) = \phi_0 + \omega t,$$

where $\phi_0 = \phi(t = 0)$. Likewise, for a body rotating with constant angular acceleration, α, the angular velocity increases linearly with time, so that

$$\omega(t) = \omega_0 + \alpha t,$$

and the rotation angle satisfies

$$\phi(t) = \phi_0 + \omega_0 t + \frac{1}{2} \alpha t^2.$$

$$\omega_0 = \omega(t = 0)$$

Note that the rotation angle plays the role of displacement, angular velocity plays the role of (regular) velocity, and angular acceleration plays the role of (regular) acceleration.

Moment of inertia

We will leave it to your physics class to really explain what moment of inertia means. Very briefly it measures an object's resistance (inertia) to a change in its rotational motion. It is analogous to the way mass measure the resistance to changes in the object's linear motion.

Because it has to do with rotational motion the moment of inertia is always measured about a reference line, which is thought of as the axis of rotation.

For a point mass, m, the moment of inertia about the line is

$$I = m d^2,$$

where d is the distance from the mass to the line. (The letter I is a standard notation for moment of inertia.)

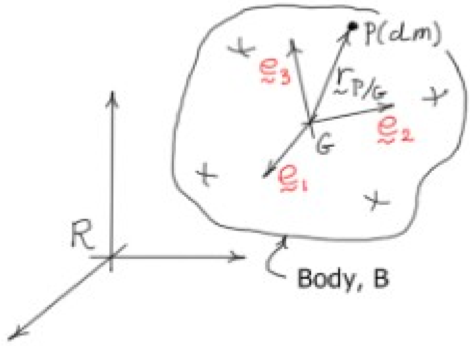
If we have a distributed mass we compute the moment of inertia by summing the contributions of each of its parts. If the mass has a continuous distribution, this sum is, of course, an integral.

A rigid body B is shown in the diagram below. The unit vectors $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ are fixed in the body and are directed along a ***convenient*** set of axes (x, y, z) that pass through the mass center G. The ***moments of inertia*** of the body about these axes are defined as follows

$$I_{xx}^G = \int_B (y^2 + z^2)\, dm$$

$$I_{yy}^G = \int_B (x^2 + z^2)\, dm$$

$$I_{zz}^G = \int_B (x^2 + y^2)\, dm$$



where x, y, and z are defined as the $\underset{\sim}{e}_i$ components of $\underset{\sim}{r}_{P/G}$ the position vector of P with respect to G, that is, $\underset{\sim}{r}_{P/G} = x\underset{\sim}{e}_1 + y\underset{\sim}{e}_2 + z\underset{\sim}{e}_3$.

Parallel Axes Theorem for Moments of Inertia

The inertia (I_i^A) of a body about an axis (i) through any point (A) is equal to the inertia (I_i^G) of the body about a parallel axis through the mass center G plus the mass (m) times the distance (d_i) between the two axes squared. Or,

$$\boxed{I_{ii}^A = I_{ii}^G + m\, d_i^2 \quad (i = x, y, \text{or } z)}$$

Note that moments of inertia are ***always positive***. From the parallel axes theorem, it is obvious that the ***minimum moments of inertia*** of a body occur about axes that pass through its ***mass center***.

Moments of inertia of a body about a particular axis measure the distribution of the body's mass about that axis. The smaller the inertia the more the mass is concentrated about the axis. Inertia values can be found either by measurement or by calculation. Calculations are based on direct integration or on the "body build-up" technique. In the body build-up technique, inertias of simple shapes are added to estimate the inertia of a composite shape.These values are transferred to axes through the composite mass center using the Parallel Axes Theorem for Moments of Inertia.

Products of Inertia

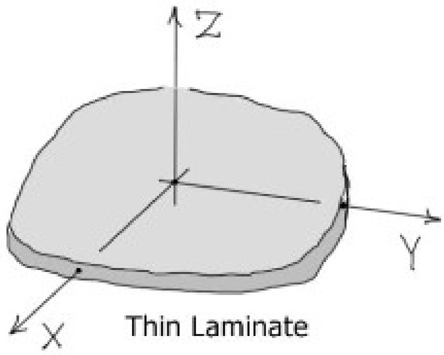
The ***products of inertia*** of the rigid body are defined as

$$I_{xy}^{G} = \int_{B} (xy)\, dm \qquad I_{xz}^{G} = \int_{B} (xz)\, dm \qquad I_{yz}^{G} = \int_{B} (yz)\, dm$$

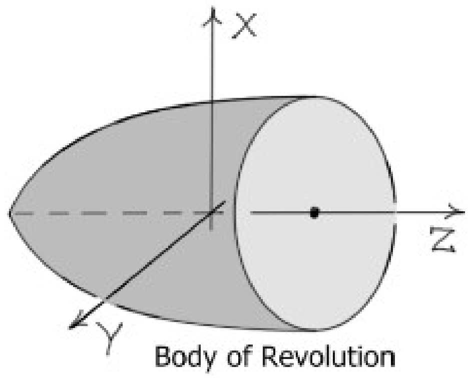
The products of inertia of a body are measures of ***symmetry***. ***If a particular plane is a plane of symmetry, then the products of inertia associated with any axis perpendicular to that plane are zero***. For example, consider the ***thin laminate*** shown. The middle plane of the laminate lies in the *XY*-plane so that half its thickness is above the plane and half is below. Hence, the ***XY-plane*** is a ***plane of symmetry*** and

$$I_{xz} = I_{yz} = 0$$

Bodies of revolution have ***two planes*** of ***symmetry***. For the configuration shown, the *XZ* and *YZ* planes are planes of symmetry. Hence, ***all products of inertia are zero*** about the *X*, *Y*, and *Z* axes.



Thin Laminate



Body of Revolution

Products of inertia are found either by measurement or by calculation. Calculations arebased on direct integrationor on the "body build-up" technique. In the body build-up technique, products of inertia of simple shapes are added to estimate the products of inertia of a composite shape. The products of inertia of simple shapes (about their individual mass centers)are found in standard inertia tables. These values are transferred to axes through the composite mass center using the

Parallel Axes Theorem for Products of Inertia

Parallel Axes Theorem for Products of Inertia

The product of inertia (I_{ij}^{A}) of a body about a pair of axes (i, j) passing through any point (A) is equal to the product of inertia (I_{ij}^{G}) of the body about a set of parallel axes through the mass center G plus the mass (m) times the product of the coordinates $(c_i c_j)$ of G relative to A measured along those axes.

$$\boxed{I_{ij}^{A} = I_{ij}^{G} + m\, c_i\, c_j \qquad (i = x, y, \text{ or } z \;\; and \;\; j = x, y, \text{ or } z)}$$

Products of inertia may be ***positive***, ***negative***, or ***zero***.

Euler Equations.

The Fundamental equation of a rotating body is

$$\mathbf{T} = \frac{d\mathbf{L}}{dt}, \qquad 1$$

is only valid in an *inertial* frame. However, we have seen that $\mathbf{L}$ is most simply expressed in a frame of reference whose axes are aligned along the principal axes of rotation of the body. Such a frame of reference *rotates* with the body, and is, therefore, *non-inertial*. Thus, it is helpful to define *two* Cartesian coordinate systems, with the same origins. The first, with coordinates x, y, z, is a fixed inertial frame--let us denote this the *fixed frame*. The second, with coordinates x', y', z', co-rotates with the body in such a manner that the x'-, y'-, and z'-axes are always pointing along its principal axes of rotation--we shall refer to this as the *body frame*. Since the body frame co-rotates with the body, its instantaneous angular velocity is the same as that of the body. Hence, it follows from the analysis

$$\frac{dL}{dt} = \frac{dL}{dt'} + \omega \times L. \qquad 2$$

Here, $\mathbf{d/dt}$ is the time derivative in the fixed frame, and $\mathbf{d/dt'}$ the time derivative in the body frame. Combining Equations (1) and (2), we obtain

$$\mathbf{T} = \frac{dL}{dt'} + \omega \times L. \qquad 3$$

Now, in the body frame let $\mathbf{T} = (T_{x'}, T_{y'}, T_{z'})$ and $\boldsymbol{\omega} = (\omega_{x'}, \omega_{y'}, \omega_{z'})$. It follows that $\mathbf{L} = (I_{x'x'}\,\omega_{x'}, I_{y'y'}\,\omega_{y'}, I_{z'z'}\,\omega_{z'})$, where $I_{x'x'}$, $I_{y'y'}$ and $I_{z'z'}$ are the principal moments of inertia. Hence, in the body frame, the components of Equation (3) yield

$$\begin{aligned}
T_{x'} &= I_{x'x'}\,\dot{\omega}_{x'} - (I_{y'y'} - I_{z'z'})\,\omega_{y'}\,\omega_{z'}, \\
T_{y'} &= I_{y'y'}\,\dot{\omega}_{y'} - (I_{z'z'} - I_{x'x'})\,\omega_{z'}\,\omega_{x'}, \\
T_{z'} &= I_{z'z'}\,\dot{\omega}_{z'} - (I_{x'x'} - I_{y'y'})\,\omega_{x'}\,\omega_{y'},
\end{aligned} \qquad 4$$

where $\dot{} = \mathbf{d/dt'}$. Here, we have made use of the fact that the moments of inertia of a rigid body are *constant* in time in the co-rotating body frame. The above equations are known as *Euler's equations.*

Consider a rigid body which is constrained to rotate about a fixed axis with *constant* angular velocity. It follows that $\dot{\omega}_{x'} = \dot{\omega}_{y'} = \dot{\omega}_{z'} = 0$.

Hence, Euler's equations (4), reduce to

$$\begin{aligned} T_{x'} &= -(I_{y'y'} - I_{z'z'})\,\omega_{y'}\,\omega_{z'}, \\ T_{y'} &= -(I_{z'z'} - I_{x'x'})\,\omega_{z'}\,\omega_{x'}, \\ T_{z'} &= -(I_{x'x'} - I_{y'y'})\,\omega_{x'}\,\omega_{y'}. \end{aligned} \qquad 5$$

These equations specify the components of the steady (in the body frame) torque exerted on the body by the constraining supports.

Applications of Euler Equations

1 Torque-free motion of a symmetric rigid body

Now consider the case when two of the moments of inertia are equal. This happens when th rigid body is rotationally symmetric around one axis. Let the z-axis be the axis of symmetry. Then $I_1 = I_2$, and the torque-free Euler equations become

$$\begin{aligned} 0 &= I_1\dot{\omega}_1 - \omega_2\omega_3\,(I_1 - I_3) \\ 0 &= I_1\dot{\omega}_2 + \omega_1\omega_3\,(I_1 - I_3) \\ 0 &= I_3\dot{\omega}_3 \end{aligned}$$

The final equation shows that ω_3 is constant. Defining the constant frequency

$$\Omega \equiv \omega_3 \left(\frac{I_1 - I_3}{I_1}\right)$$

$$\omega_2 = -A\sin\Omega t + B\cos\Omega t$$

Notice that

$$\omega_1^2 + \omega_2^2 = A^2 + B^2$$

so the x and y components of the angular velocity together form a constant length vector that precesses around the z axis. If the angular velocity is dominated by ω_3, the remaining components give the object a "wobble" – it spins slightly off its symmetry axis, precessing. On the other hand, if ω_3 is small, the motion is a "tumble" – end over end rotation *of* its symmetry axis.

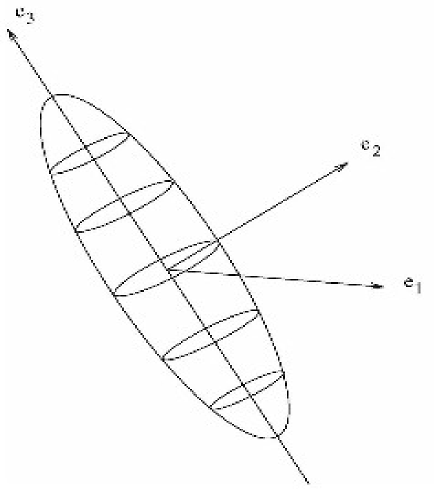
Remember that this analysis takes place in a frame of reference rotating with angular velocity $\boldsymbol{\omega}$. If all of the motion were about the z-axis, the object would be at rest in the rotating frame. The fact that we get time dependence of our solution for $\boldsymbol{\omega}$ means that even in a frame rotating with the body, the body precesses. If we transform back to the inertial frame, it is also spinning.

$$\omega_1 = A\cos\Omega t + B\sin\Omega t$$

for ω_1 and, returning to the original equation $\dot{\omega}_1 = \Omega\omega_2$,

2 The Symmetric Top

The symmetric top is an object with $I_1 = I_2 \neq I_3$ The typical figure below explains this;



Euler equations become

$$\begin{aligned}
I_1\dot{\omega}_1 &= \omega_2\omega_3(I_1 - I_3)\\
I_2\dot{\omega}_2 &= -\omega_1\omega_3(I_1 - I_3)\\
I_3\dot{\omega}_3 &= 0
\end{aligned}$$

So, in this case, we see that ω_3 ,which is the spin about the symmetric axis, is a constant of motion. In contrast, the spins about the other two axes are time dependant and satisfy

$$\dot{\omega}_1 = \Omega\omega_2 \quad , \quad \dot{\omega}_2 = -\Omega\omega_1$$

where

$$\Omega = \omega_3(I_1 - I_3)/I_1$$

is a constant. These equations are solved by

$$(\omega_1, \omega_2) = \omega_0(\sin\Omega t, \cos\Omega t)$$

for any constant ω_0. This means that, in the body frame, the direction of the spin is not constant: it *precesses* about the $\mathbf{e}_3$ axis with frequency Ω. The direction of the spin depends on the sign on Ω or, in other words, whether $I_1 > I_3$ or $I_1 < I_3$. This is drawn in figure below; In an inertial frame, this precession of the spin looks like a wobble.

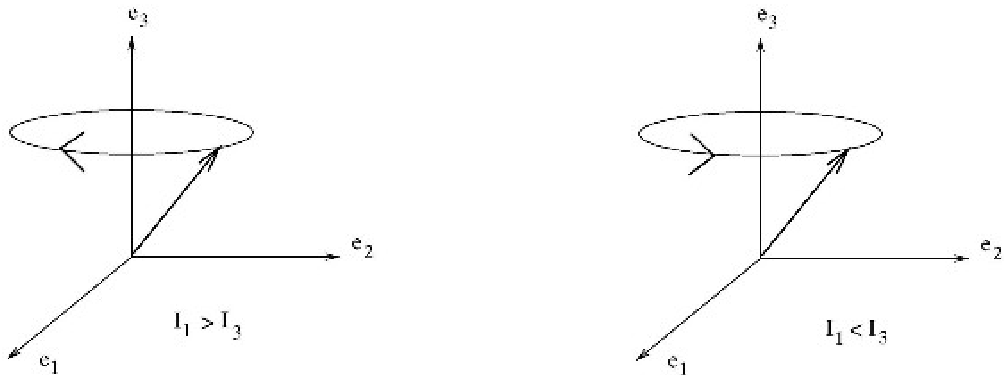


Fig:*The precession of the spin: the direction of precession depends on whether the object is short and fat ($I_3 > I_1$) or tall and skinny ($I_3 < I_1$)*

Moment of Inertia Tensor.
Consider a rigid body rotating with fixed angular velocity $\boldsymbol{\omega}$ about an axis which passes through the origin--see Figure below. Let $\mathbf{r}_i$ be the position vector of the ith mass element, whose mass is m_i. We expect this position vector to *precess* about the axis of rotation (which is parallel to $\boldsymbol{\omega}$) with angular velocity $\boldsymbol{\omega}$. It, therefore, that

$$\frac{d\mathbf{r}_i}{dt} = \boldsymbol{\omega} \times \mathbf{r}_i.$$

As, $v = r * \omega$
Thus, the above equation specifies the velocity, $\mathbf{v}_i = d\mathbf{r}_i/dt$, of each mass element as the body rotates with fixed angular velocity about an axis passing through the origin.

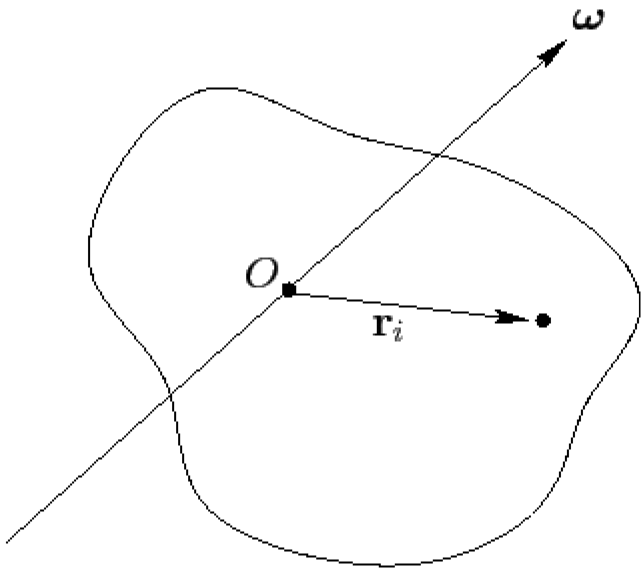


Fig: *A rigid rotating body.*
The total angular momentum of the body (about the origin) is written

$$\mathbf{L} = \sum_{i=1,N} m_i \, \mathbf{r}_i \times \frac{d\mathbf{r}_i}{dt} = \sum_{i=1,N} m_i \, \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) = \sum_{i=1,N} m_i \left[r_i^2 \, \boldsymbol{\omega} - (\mathbf{r}_i \cdot \boldsymbol{\omega}) \, \mathbf{r}_i \right],$$

where use has been made of above Equation, and some standard vector identities. The above formula can be written as a matrix equation of the form

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix},$$

Here, I_{xx} is called the *moment of inertia* about the x-axis, I_{yy} the moment of inertia about the y-axis, I_{xy} the xy *product of inertia*, I_{yz} the yz product of inertia, *etc.* The matrix of the I_{ij} values is known as the *moment of inertia tensor*. Note that each component of the moment of inertia tensor can be written as either a sum over separate mass elements, or as an integral over infinitesimal mass elements

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Thanks

UNIT III

Simple Harmonic Motion

When a body moves periodically in a straight line on either side of a point, the motion of body is called the simple harmonic motion. Thus, the simple harmonic motion is a special case of the periodic motion, obviously, a simple harmonic motion is definitely a periodic motion, but all the periodic motions are not the simple harmonic motions.

For example, the motion of earth around the sun is a periodic motion, but it is not the simple harmonic motion. On the other hand, the motion of a simple pendulum is simple harmonic motion as well as the periodic.

In a simple harmonic motion, the body moves periodically in a straight line on either side of its an position such that its acceleration is proportional to the displacement of the particle and the direction of acceleration is always towards the mean position. In other words, the motion of a body under a restoring force is a simple harmonic motion.

Restoring Force

The force, which is directly proportional to the displacement of the body from its mean position and is directed towards the mean position is called the restoring force i.e., the restoring force tends to bring the body back to its mean position.

Conditions or Characteristics of Simple Harmonic Motion

Following are the conditions (or characteristics) of simple harmonic motion:

(i)The motion must be in straight line on either side of a definite point (mean position).

(ii)The moving body must pass from its mean position repeatedly after a definite time i.e., motion must be periodic.

(iii) The acceleration of the moving body must always be proportional to the displacement of the body from its mean position and the direction of acceleration must always be towards the mean position, i.e.Acceleration α displacement and in the direction opposite to displacement.

If at any instant, the displacement of body from its mean position is x, the acceleration of the body is

$$a \propto -x$$

But by Newton's law of motion, force = mass x acceleration

$$F \propto -x$$

Thus, the force acting on the body must be proportional to the displacement of the body from its mean position and its direction must be towards the mean position (i.e., the force must be the restoring force).

A system, whose motion is simple harmonic, is known as the simple harmonic oscillator.

For example, motion of simple pendulum for small amplitude, motion of the compound pendulum, motion of the mass attached at the free end of a spring rigidly fixed at its other end, motion of torsional pendulum etc.

Harmonic Oscillator

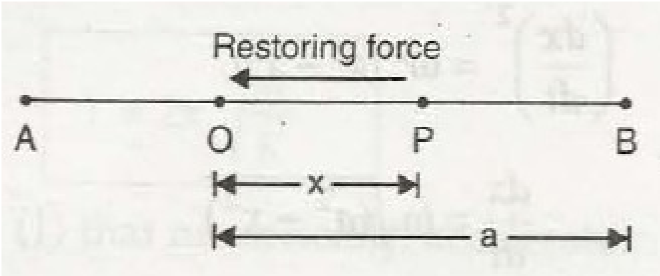
A system executing simple harmonic motion is called a simple harmonic oscillator. In simple harmonic motion, the force acting on the system at any instant, is directly proportional to the displacement from a fixed point in its path and the direction of this force is towards that fixed point. Thus, the system executes the motion under a linear restoring force. If the displacement of the system from a fixed point is x, the linear restoring force is -Kx, where K is a constant which is called the force constant. Thus no other force except the linear restoring force acts on a simple harmonic oscillator. As a result, the oscillator executes vibrations of constant amplitude and with a constant frequency. These oscillations are called the free oscillations.

Let a particle of mass m be executing simple harmonic oscillations. The acceleration of the particle at displacement x from a fixed point will be d^2x/dt^2 . For the particle, $Restoring Force \propto -displacement$

$$m \frac{d^2x}{dt^2} \propto -x \quad \text{or} \quad m \frac{d^2x}{dt^2} = -Kx$$

where K is a constant, which is called force constant of the particle. Here the negative sign tells that the direction of force acting on the particle (or acceleration) is opposite to the direction of increase in displacement.

See figure below



Acce1eration of the particle $\frac{d^2x}{dt^2} = \frac{-Kx}{m}$

Let $\frac{K}{m} = w^2$ then,

Acceleration of the particle $\frac{d^2x}{dt^2} = -w^2x$ 1

Equation (1) is known as differential equation of simple harmonic oscillator.

Solution of Differential Equation of Simple Harmonic Oscillator

Now we have to find the displacement x of the particle at any instant t by solving the differential equation (1) of the simple harmonic oscillator.

In equation (1), multiplying by 2(dx/dt),we get

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} + \omega^2 x \,.\, 2\frac{dx}{dt} = 0$$

$$\frac{d}{dt}\left(\frac{dx}{dt}\right)^2 + \omega^2 \frac{d}{dt}(x^2) = 0$$

On integrating $\left(\frac{dx}{dt}\right)^2 + \omega^2x^2 = A$ (constant)

At the position of maximum displacement, i. e., at x =±a, ve1ocity of particle dx/dt = 0

$0 + w^2a^2 = A$

Then $\quad (\frac{dx^2}{dt^2}) + w^2x^2 = w^2a^2$

Hence $\quad (\frac{dx}{dt})^2 = w^2(a^2 - x^2)$

or velocity of the particle $$\frac{dx}{dt} = w\sqrt(a^2 - x^2) \qquad 2$$

Equation (2) tells us the velocity of particle at position x

From equation (2) $\quad \frac{dx}{\sqrt(a^2 - x^2)} = wdt$

Again integrating, $\quad sin^{-1}\frac{x}{a} = wt + \phi(constant)$

$$x = asin(wt + \phi) \qquad 3$$

Here a is the amplitude of oscillations and φ is the initial phase of the motion of particle (whose value can be known from the initial conditions).

(i) If x = 0 at t = 0 (i.e. the particle_ initiates its oscillations from its mean position), then sin φ = 0 or φ = 0, then the displacement equation of the particle executing simple harmonic motion at any time t will be

x =a sin ωt

(ii) If x = 0 at t = 0 (i.e. , the particle initiates its oscillations from its maximum displaced position), then sin φ = 0 or φ = 2π, then the displacement equation of the particle executing simple harmonic motion at any instant t will be

$$x = asin(wt + 2\pi) = acoswt \qquad 4$$

Time Period and Frequency

It is clear from equations (4) and (5), that

$asinw(t + 2\pi/w) = asinwt$ **and** $acosw(t + 2\pi/w) = acoswt$

It is concluded from here that the displacement of the particle at any instant (t + 2π/ω) is exactly the same as it was at the instant t i.e., the particle comes back to its initial position during its motion exactly after time it . Hence time period of the particle executing simple harmonic motion is

$$\boxed{T = 2\pi\sqrt{\frac{m}{K}}}$$

It is clear from equation (1) that numerically,

acceleration = $-w^2x$

$$w = \sqrt{\frac{accelration}{displacement}}$$

and time period $T = 2\pi\sqrt{\frac{displacement}{accelration}}$

Since frequency $\nu = 1/T$, hence frequency $\nu = 1/2\pi\sqrt{\frac{accelration}{displacement}}$

Remember that the quantity $w = 2/T$ is known as the angular frequency of motion .

Damped harmonic oscillator

In a simple harmonic oscillator, no other force except the linear restoring force acts. As a result, the oscillator executes vibrations of constant amplitude and with a constant frequency these oscillations are called the free oscillations. The total mechanical energy (i.e., kinetic energy +

potential energy) of the oscillator always remains conserved in such oscillations.

But in practice, free oscillations are not possible. This is because the medium in which the oscillator executes vibrations, exerts a frictional or viscous force on the oscillator. This force is called the damping force. This force acts in direction opposite to the direction of motion due to which the amplitude of vibration gradually decreases. Hence, the energy of oscillator also decreases. Such a vibrating system is called damped harmonic oscillator. For example, the simple pendulum executing oscillations in air or in any other medium, tuning fork, ballistic galvanometer are the damped harmonic oscillators.

Thus, a particle executing damped harmonic motion in a medium (i.e., damped harmonic oscillator) is acted upon by two forces:

(i) Restoring force which is directly proportional to the displacement from a fixed point on its path and is in a direction opposite to the displacement, i.e., -Kx where K is force constant and x is the displacement, is the displacement of the particle at any instant.

(ii) Damping force, which is proportional to the velocity of the particle and is in a direction opposite to motion, i.e., =r[dx/dt] , where r is a positive constant, which is called the damping coefficient.

If m is the mass of the particle, by Newton's law, the total force acting on the particle= md^2x/dt^2 where d^2x/dt^2 acceleration of the particle.

Thus the equation of motion of a damped harmonic oscillator will be

$$m\frac{d^2x}{dt^2} = -r\frac{dx}{dt} - Kx \quad \text{or} \quad \frac{d^2x}{dt^2} + \frac{r}{m}\frac{dx}{dt} + \frac{K}{m}x = 0$$

$$\boxed{\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + n^2x = 0} \qquad 1$$

$$2b = \frac{r}{m} \quad \text{and} \quad n^2 = K/m.$$

It must be noted that unit of damping coefficient r is kg/ sec. Therefore, the unit of r/ m or 2b) is s^{-1}. Thus the unit of m/r is same as that of the time. Hence the quantity m/r (or l/2b) is called the time constant or the relaxation time. It is represented by t (Tau). Relaxation time is defined as the time in which the energy of particle reduces to 0.37 times its maximum initial energy.

Now, we are to find the solution of equation (1) for damped motion. Let the solution of equation (1) be $x = Ae^{at}$

$$\frac{dx}{dt} = A\,\alpha\, e^{\alpha t} \quad \text{and} \quad \frac{d^2x}{dt^2} = A\,\alpha^2 e^{\alpha t} \qquad 2$$

Substituting these values in equation (1)

$$A\alpha^2 e^{\alpha t} + 2b\,A\,\alpha e^{\alpha t} + n^2 A\, e^{\alpha t} = 0$$

$$(\alpha^2 + 2b\alpha + n^2)\,Ae^{\alpha t} = 0 \quad \text{or} \quad \alpha^2 + 2b\alpha + n^2 = 0 \qquad 3$$

$$\alpha = -b \pm \sqrt{b^2 - n^2} = -b \pm p, \quad \text{where} \quad p = \sqrt{(b^2 - n^2)}$$

Thus the two possible solutions of equation (1) are

$x = A_1\, e^{(-b + p)t}$ and $x = A_2\, e^{(-b-p)t}$

The general solution of equation (1) is

$x = A_1\, e^{(-b + p)t} + A_2 e^{(-b-p)t}$ 4

$=e^{-bt} [A_1 e^{Pt} + A_2 e^{-Pt}]$ 5

Here, A_1 and A_2 are the constants whose values can be obtained from the initial condition of motion. There are following three cases possible:

1. When b < n, under damped.
2. When b > n, over damped.
3. When b = n, critically damped.

Case 1. Under-damped Case: If the damping is so small that b <<n or r /2m<< √ K/m, then, $p = \sqrt{(b^2 - n^2)} = \sqrt{-(n^2 - b^2)} = i\sqrt{(n^2 - b^2)}$ or p = *j*w, where w = √(n2- b2). And $i^2 = -1$

Obviously, p is an imaginary quantity, but w is a real quantity.

Then from equation (2)

$$x = e^{-bt}[A_1 e^{iwt} + A_2 e^{-iwt}]$$
$$= e^{-bt}[(A_1 + A^2)coswt + i(A_1 - A_2)sinwt]$$
$$= e^{-bt}[A_1(coswt + isinwt) + A^2(coswt - isinwt)] \quad 6$$

Since x is a real quantity, therefore (A_1+A_2) and $i(A_1-A_2)$ must also be real quantities. But A_1 and A_2 are the complex quantities, therefore

$(A_1+A_2) = a_0 \sin \varphi$ and $i(A^1 - A^2) = a_0 \cos\varphi$ in above .equation (3)

$x = {}^{-bt} [a_0 \sin \varphi \cos wt + a_0 \cos \varphi \sin wt]$

$x = a_o e^{-bt} \sin (wt+\varphi)$ 7

This expression represents the damped harmonic motion where amplitude decreases exponentially with time. Hence, it is not a simple harmonic motion but it is an oscillatory motion, whose angular frequency is

$$\omega = \sqrt{(n^2 - b^2)} = \sqrt{\left(\frac{K}{m} - \frac{r^2}{4m^2}\right)}$$

$$\text{time period } T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{(n^2 - b^2)}} = \frac{2\pi}{\sqrt{\left(\frac{K}{m} - \frac{r^2}{4m^2}\right)}} \qquad 8$$

The amplitude of motion is $a = a_0 e^{-bt} = a_0 e^{-1/2T}$

The displacement-time graph for the damped harmonic oscillations is shown in Figure below. It is evident that the amplitude of motion decreases exponentially.

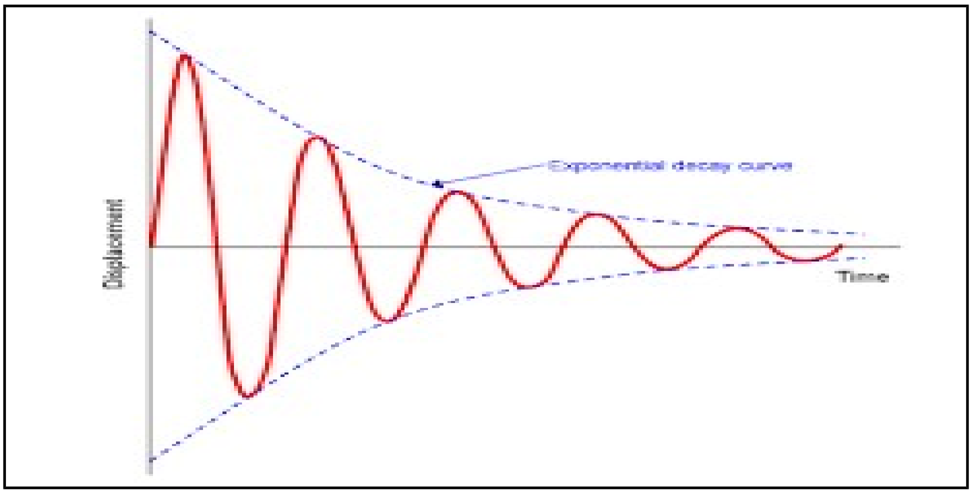
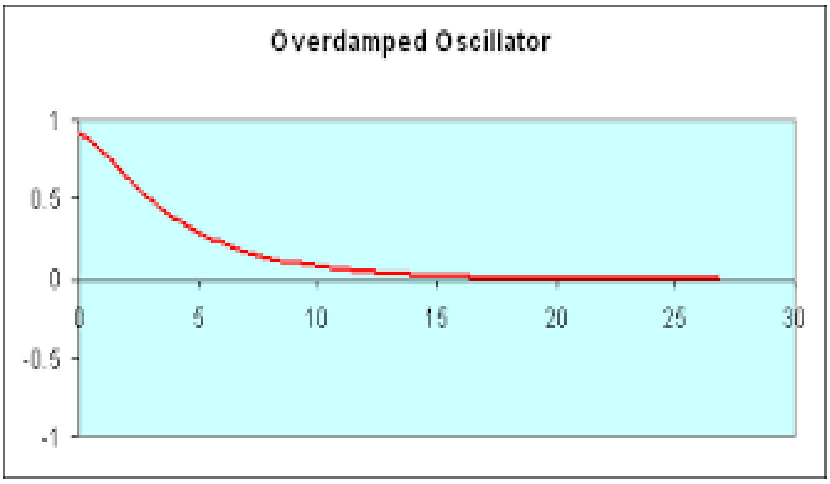


FIGURE: Displacement-time graph for the damped harmonic oscillations

Case 2. Over-damped Case: If the damping is too large that b > n, then p = √(b2- n2) is a real quantity and p <b.

Then from equation (4) $x=a_1e^{-(b-p)t}+a_2e^{-(b+p)t}$ 9

Since, p < b, hence' both the quantities on RHS of above equation decrease exponentially with time and there will no oscillations (See Figure below).



Such a motion is called the dead beat or a periodic motion. This type of motion is used in the dead beat galvanometer.

Case 3. Critically Damped Case: If b = n, then p = 0

From equation (4)

$$x = (A_1 + A_2)\, e^{-bt} = C\, e^{-bt,} \qquad 10$$

Where A_1+A_2 =c (a contant)

Since, there is only one constant in above equation, therefore it cannot be the solution of second order differential equation (1)

Now if we consider that $\sqrt{(b^2 - n^2)}$ =p a very small quantity, then

$$x = e^{-bt}\,[\, A_1\, e^{Pt} + A_2\, e^{-pt}]$$

$$= e^{-bt}\,[A_1\,(1 + pt + \ldots\,) + A_2\,(1- pt + \ldots\,)]$$

Since, p is very small, therefore neglecting the terms p^2 and p^3, ...

$$x = e^{-bt}\,[(A_1+A_2)\ + p\,(A_1 - A_2)t] = e^{-bt}\,[P + Qt] \qquad 11$$

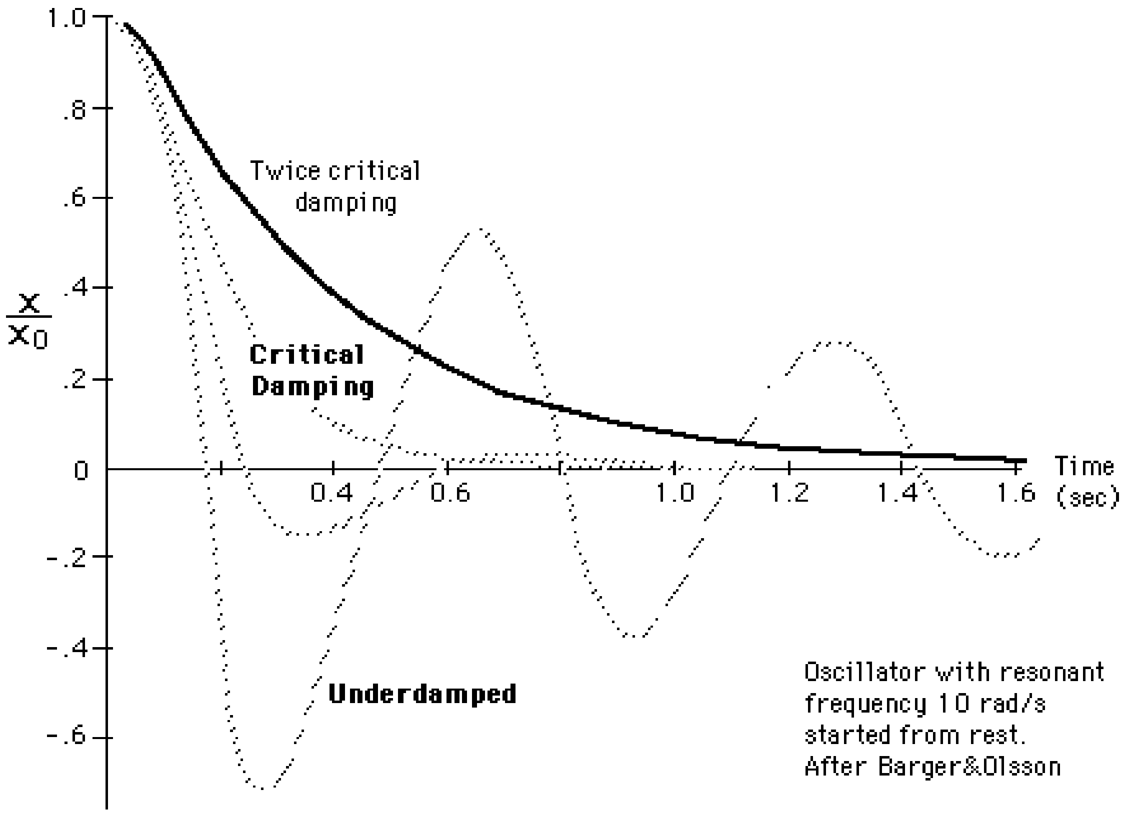
where, $A_1 + A_2 = P$ and $p[A_1 + A_2] = Q$

If initially at t= 0, the displacement of particle is x= x0 and velocity $v = dx/dt = V_0$, then at $t = 0$, $x_0 = P$ and from

$dx/dt = Qe^{-bt} - be^{-bt}(P + Qt)$ at $t = 0$, $V_0 = Q - bP$.

Hence, $Q = v_0 + b_{x0}$

Then, $x = [x_0 + (v_0 + b_{xo})\,t]\,e^{-bt}$ 12



In the above expression, due to the coefficient $[x_0 + (v_0 + bx_0)t]$, first the value of x increases with increase in t, but later on the exponential term e^{-bt} becomes more pronounced due to which the value of x falls to zero in a small time interval and no oscillations occur.

Power Dissipation in Damped Harmonic Oscillator

We have read that in damped oscillations, the mechanical energy of the particle continuously decreases due to the damping forces. As a result, the amplitude of motion also decreases with time. The displacement of particle executing damped harmonic motion at any instant tis

$x = a_0 e^{-bt} \sin(wt+\varphi)$ from education

Velocity of the particle $V = dx/dt$

$= a_0 e^{-bt} [(-b) \sin(wt+\varphi) + w \cos(wt+\varphi)]$

If m is the mass of the particle, the kinetic energy of the particle is

$$K = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2$$
$$= \frac{1}{2} m a_0^2 e^{-2bt} [b^2 \sin^2(\omega t+\phi) + \omega^2 \cos^2(\omega t+\phi) - 2b\omega \sin(\omega t+\phi) \cos(\omega t+\phi)] \quad 13$$

And potential energy

$$U = \int_0^x m\omega^2 x \, dx = \frac{m\omega^2 x^2}{2} \quad 14$$
$$= \frac{1}{2} m \omega^2 a_0^2 e^{-2bt} \sin^2(\omega t+\phi)$$

Average total energy of oscillator in a periodic time = Average kinetic energy + Average potential energy

Now, to evaluate the average kinetic energy and the average potential energy in one time period, we can assume that the amplitude of oscillation remains nearly unchanged i.e.,

e^{-2bt} remains nearly constant. Since the average value of $\sin^2(wt+\varphi)$ and $\cos^2(wt+\varphi)$ is 1/2 for one time period and average value of

$\sin(wt+\varphi)\cos(wt+\varphi)$ is zero over one time period, therefore in a period of time.

Average Kinetic energy =

$$= \frac{1}{2} m a_0^2 e^{-2bt} \times \left(\frac{b^2+\omega^2}{2}\right) = \frac{1}{4} m a_0^2 \omega^2 e^{-2bt} \quad 15$$

And the average potential energy

$$= \frac{1}{2} m a_0^2 \omega^2 e^{-2bt} \times \frac{1}{2} = \frac{1}{4} m a_0^2 \omega^2 e^{-2bt} \qquad 16$$

Hence, the average total energy in one periodic time

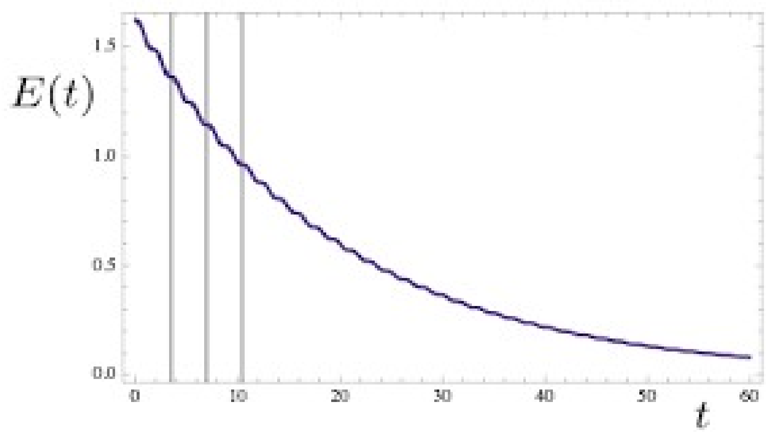
$$E_{av} = \frac{1}{4} m a_0^2 \omega^2 e^{-2bt} + \frac{1}{4} m a_0^2 \omega^2 e^{-2bt}$$

$$= \frac{1}{2} m a_0^2 \omega^2 e^{-2bt} = \frac{1}{2} m a_0^2 \omega^2 e^{-t/\tau}$$

$$E_{av} = E_0 \, e^{-2bt}$$

$$E_0 = \frac{1}{2} m a_0^2 \omega^2 \qquad 17$$

It is clear from above equation that the average energy of system in damped oscillations decreases exponentially with time as shown in Figure below.



The rate of loss of energy or the average power loss at any instant

Clearly, the rate of loss of energy at any instant depends average energy with time in damped oscillator on the damping of the system (more the value of b, more is the rate of loss in energy).

This loss in energy generally appears in the form of heat energy of the oscillating system.

Quality Factor of Damped Harmonic Oscillator

The quality factor of an oscillator expresses its efficiency. It is represented by the letter Q. Since there is no energy loss in free oscillations of an ideal oscillator, its efficiency is maximum.

Definition: The quality factor of an oscillator is defined as the product of 2π with the ratio of the average energy stored in the oscillator at any instant to the loss in energy in one periodic time.

i.e, Quality factor

Q =2π * energy stored at any instant / energy lost in one second

But from equation (17)

$$Q = \frac{2\pi \times E_{av}}{PT} = \frac{E_{av}\,\omega}{P} \qquad \left[\because \ \frac{2\pi}{T} = \omega\right]$$

$$P = E_{av}/\tau \quad \text{or} \quad E_{av}/P = \tau$$

$$\boxed{Q = \omega\tau}$$

Thus, quality factor is a dimensionless quantity. More the relaxation time of an oscillator, more is its quality factor

Driven Harmonic Oscillator

When an external periodic force is applied on a system, the force imports a periodic pulse to the system so that the loss in energy in doing work against the dissipative forces is recovered.

As a result, the system is continuously oscillates. In the initial stages, the system tends to execute oscillations with the natural frequency (or frequency of free vibrations), while the impressed periodic force tries to impose its own frequency on it. Therefore, the free vibrations of the body soon die out and ultimately, the system starts oscillating with a constant amplitude and with the frequency equal to that of the impressed force. This is called the steady state of an oscillator. These vibrations are called the forced vibrations. The force impressed on the system is called the driver and the system which executes forced vibrations, is called the forced or driven harmonic oscillator.

Thus a particle executing the forced harmonic oscillations is acted upon by the following three forces:

(i) A linear restoring force($= -Kx$), which is directly proportional to the displacement from a fixed point and is in a direction opposite to the displacement.

(ii) A damping force ($=- r\, dx/dt$),which is directly proportional to the velocity and is in direction opposite to the motion.

An external periodic force($F=F0 \sin \omega t$)where F0 is the amplitude of the impressed force and w is the angular frequency of the impressed force.

If m be the mass of particle executing forced oscillations and d^2x/dt^2: is its acceleration at dt any instant, then by Newton's law.

Total force acting on the particle = md^2x/dt^2

The equation of motion of forced harmonic oscillations will be

$$m\frac{d^2x}{dt^2} = -Kx - r\frac{dx}{dt} + F$$

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + Kx = F_0 \sin \omega t$$

$$\frac{d^2x}{dt^2} + \frac{r}{m}\frac{dx}{dt} + \frac{K}{m}x = \frac{F_0}{m}\sin \omega t \qquad 1$$

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + n^2x = f \sin \omega t$$

where, $r/m = 2b$, $K/m= n^2$ and $F_0/m = f$

Obviously from equation (1), the differential equation for the free oscillations of the system will be $d^2x + n^2x = 0$,

where $n = \sqrt{K/m}$ =angular frequency of free oscillations.

Transient and Steady States

We have read that on applying an external periodic force, initially the system tends to oscillate with an angular frequency $w_1 = \sqrt{(n^2-b^2)}$ due to damping, but the driving force of angular frequency w acting on the system forces it to oscillate with its own frequency. In this way, the actual motion of system is obtained by the superposition of two oscillations, whose angular frequencies are $w_1 = \sqrt{(n^2-b^2)}$ and $w_2 = w$ respectively. Hence if $w \neq n$, the solution of the equation (1) can be written as follows

$x = x_1 + x_2$

where, x_1 is the solution of equation (1), when the external force is zero. Then from equation (1)

$$\frac{d^2x_1}{dt^2} + 2b\frac{dx_1}{dt} + n^2x_1 = 0 \qquad 2$$

$$\frac{d^2x_2}{dt^2} + 2b\frac{dx_2}{dt} + n^2x_2 = 0 = f sinwt \qquad 3$$

Remember that in the complete solution $x = x_1 + x_2$, x_1 is known as the complementary function and x_2 is known as the particular integral.

The complementary function represented by equation (2) decreases exponentially with time and after some time, this term vanishes, hence it is also known as the transient solution of the forced harmonic oscillator. In this way, the system in transient state, oscillates with a frequency different from the natural frequency or the frequency of the driving force.

After a long time, when t >> t, the natural oscillations of the system vanishes due to damping and then the system oscillates with the frequency of the driving force. This state of the system is known as the steady state.

Let the solution of equation (2) in the steady state be $x_2 = A \sin(wt-\theta)$. [Since in the steady state, the amplitude of forced oscillation is constant (=A, say) and the frequency is equal to the frequency of impressed force i.e., $\omega/2\pi$. Here θ is the phase difference between the displacement and the impressed force.Then

$dx_2/dt = A\,\omega \cos(\omega t-\theta)$ and $d^2x_2/dt^2 = -A\omega^2 \sin(\omega t-\theta)$

Substituting these values in equation (3)

$-A\omega^2 - sin(\omega t-\theta)+2bAwcos(\omega t-\theta) + n^2Asin(\omega t-\theta) = fsin(\omega t-\theta+\theta)$

or

$A(n^2-w^2)sin(\omega t-\theta)+2bAwcos(\omega t-\theta)=f[sin(\omega t-\theta)cos\theta+cos(\omega t-\theta)sin\theta]$

Since this equation is valid for all value of t, therefore by equating the coefficients of sin$(\omega\ t-\ \theta)$ and cos$(\omega\ t-\ \theta)$ separately, we get

$A(n^2-w^2)$ = fcosθ and 2bAω = fsin θ

Squaring and adding the above equations

$A^2[(n^2-w^2)^2 + 4b^2\ \omega^2\ ] = f^2[\cos_2 \theta + \sin^2 \theta]$

$$A = \frac{f}{[4b^2\omega^2 + (n^2 - \omega^2)^2]^{1/2}}$$

$$\tan\theta = \frac{f\sin\theta}{f\cos\theta} = \frac{2bA\omega}{A(n^2 - \omega^2)} = \frac{2b\omega}{n^2 - \omega^2} \quad 4$$

$$\theta = \tan^{-1}\left(\frac{2b\omega}{n^2 - \omega^2}\right)$$

Putting these values of A and θ in x_2 =Asin (ω t- θ), the solution of equation (3) is

$$\boxed{x_2 = \frac{f}{[4b^2\omega^2 + (n^2 - \omega^2)^2]^{1/2}} \sin\left[\omega t - \tan^{-1}\left(\frac{2b\omega}{n^2 - \omega^2}\right)\right]}$$

Substituting the values of x_1 and x_2; the complete solution of differential equation (1) of forced oscillator is given as

$$\boxed{x = a_0 e^{-bt}\sin(nt + \phi) + \frac{f}{[4b^2\omega^2 + (n^2 - \omega^2)^2]^{1/2}} \sin\left[\omega t - \tan^{-1}\frac{2b\omega}{n^2 - \omega^2}\right]} \quad 5$$

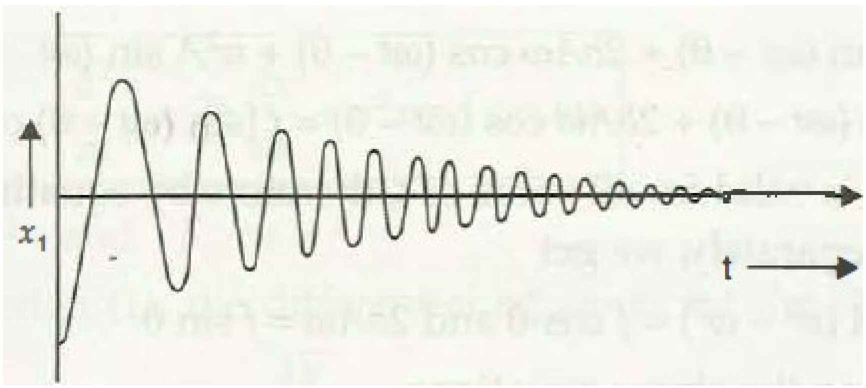
In the above equation, the first term on the RHS which is transient part, decreases with time and finally its role vanishes. The time upto which it plays its role, depends on the amount of damping. More damping, more rapidly this term decreases to zero.

When the damping is zero (i.e., when b = 0), in steady state

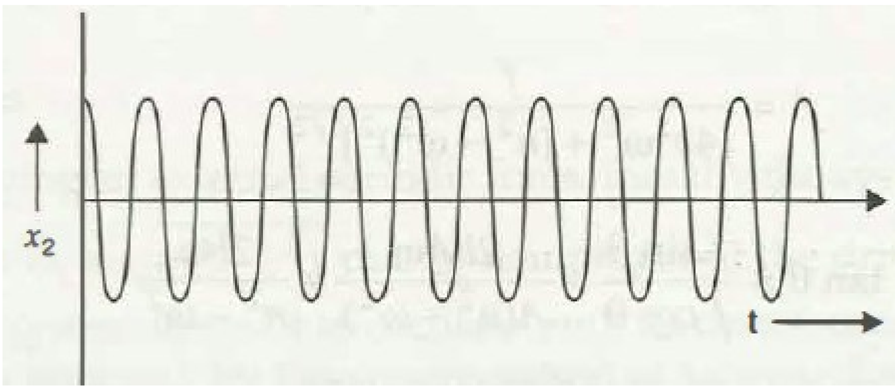
$$x = f/(n^2\text{-}w^2)\ \sin wt \qquad 6$$

and θ = 0° (i.e., the driving force and displacement will be in same phase). It is concluded that the phase difference between the displacement and driving force of a forced oscillator is due to damping. It is also clear that when w = n, the amplitude of oscillations become infinite. This condition is known as Resonance.

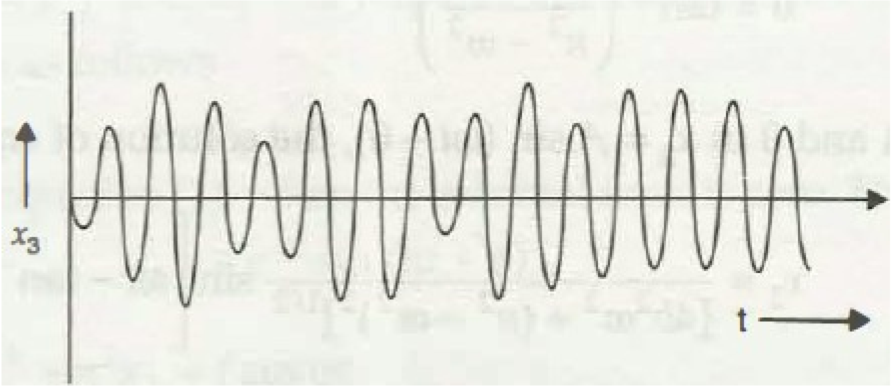
Figure below represents the variation of the transient displacement x_1 steady displacement x_2 and their sum, i.e., total displacement x with time t of a forced oscillator for low damping.



(a) Variation of transient displacement x_1 with time



(b) Variation of steady displacement x_2 with time



(c) Variation of total displacement x with time

FIGURE: Variation of displacement of a forced oscillator with time

Above equation gives the displacement of the forced oscillator at any instant t. It is clear that the amplitude of the driven oscillator in steady state does not depend on time t (i.e., it remains constant with time), but it depends on the frequency roof the external periodic force.

Now we will study the following three cases:

(i) When $\omega <<$ n, i.e., the frequency of the driving force is much less.

(ii) When ω = n, i.e., the state of resonance.

(iii) When ω > n, i.e., the frequency of the driving force is much high.

Case (i) When ω << n, i.e., the driving frequency is less than the natural frequency of the driven. For low damping (when b = 0), from equation (4) and equation (5)

$$\tan \theta = 0 \text{ or } e = \infty \qquad 7$$

Thus in this case, the displacement is in phase with the driving force and the amplitude of oscillations does not depend on the mass and damping, but only depends on the force constant.

Case (ii). When ω = n, i.e., the frequency of driver is equal to the natural frequency of the driven. Then from equation (4)

$$A = \frac{f}{2b\omega} = \frac{f\tau}{\omega} = \frac{f\tau}{n} = \frac{F_0/m}{(r/m)\,n} = \frac{F_0}{nr} \qquad 8$$

Since t = 1/2b and ω = n]

This is called the state of resonance. Thus at resonance, the amplitude of oscillation depends on the damping coefficient r. Low the damping, more is the amplitude. If damping is zero (i.e., r = 0 or 't = ∞),

$A_{max} = \infty$(infinite)

From, equation (4) tan $\theta = \infty$ or $\theta = \pi/2$

i.e., at resonance, the displacement of the oscillator lags behind the driving force in phase by $\pi/2$. Remember that the amplitude represented by equation (8) is not maximum. The reason behind it is as follows:

From, $$A = \frac{f}{\sqrt{[4b^2w^2 + (n^2 - w^2)^2]}}$$

it is clear that for A to be maximum, the value of the term

$$\frac{d}{d\omega}\left[\sqrt{(4b^2\omega^2 + (n^2 - \omega^2)^2}\right] = 0$$

$$\frac{d}{d\omega}[4b^2\omega^2 + (n^2 - \omega^2)^2] = 0$$

must be minimum. i.e.,

$8b^2\omega + 2(n^2 - \omega^2)^2 (-2\omega) = 0$

$\omega^2 = n^2 - 2b^2 = \omega_r^2$ (Say)

$$A_{max} = \frac{f}{\sqrt{(4b^2n^2 - 4b^4)}} = \frac{f}{2b\sqrt{(n^2 - b^2)}}$$

Thus, at a particular frequency of the driver, the amplitude of oscillator becomes maximum. This phenomenon is called the amplitude resonance and this particular frequency is called the resonance frequency. The resonant frequency of forced harmonic oscillator

$$f_r = \frac{\omega_r}{2\pi} = \frac{1}{2\pi}\sqrt{(n^2 - 2b^2)}$$

If damping is zero (i.e., b = 0), than ω, = n (i.e., the resonant angular frequency of the oscillator is equal to the natural angular frequency of the driven) and maximum amplitude Amax =infinite. Figure 1.15 shows the resonant amplitude of different case.When an external periodic force is applied on a system, the force imports a periodic pulse to the system so that the loss in energy in doing work against the dissipative forces is recovered

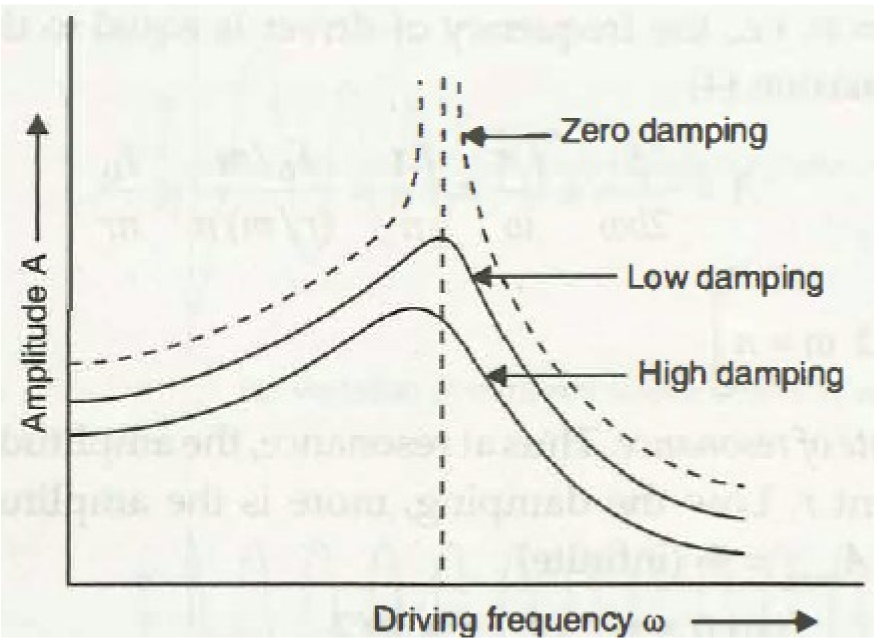


FIGURE:Variation of phase difference with frequency in steady state

Anharmonic Oscillator

In classical cases, anharmonicity is the deviation of a system from being a hormonic oscillator. An oscillator that is not oscillating in S.H.M is known as an anharmonic oscillator where the system can be approximated to a harmonic oscillator and the anharmonicity can be calculated using perturbation. If the anharmonicity is large, then it involves higher physics.

As a result, oscillations with frequencies 2ω and 3ω etc., where is the ω fundamental frequency of the oscillator, appear. Furthermore, the frequency ω_0 deviates from the frequency ω of the harmonic oscillations. As a first approximation, the frequency shift $\Delta\omega = \omega - \omega_0$ is proportional to the square of the oscillation amplitude $\boldsymbol{A}$

$$\Delta\omega \propto A^2$$

In a system of oscillators with natural frequencies $w\alpha$, $w\beta$ ... anharmonicity results in additional oscillations with frequencies .

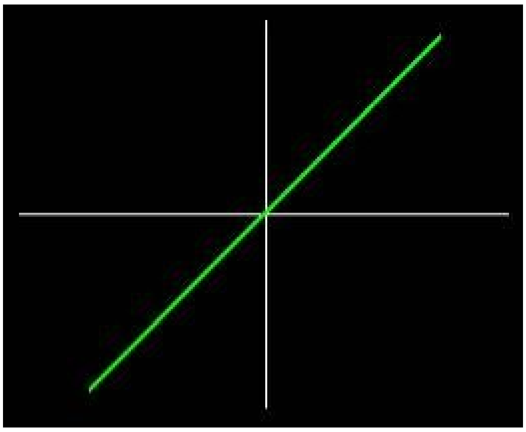
$$\omega_\alpha \pm \omega_\beta$$

Lissajous figures

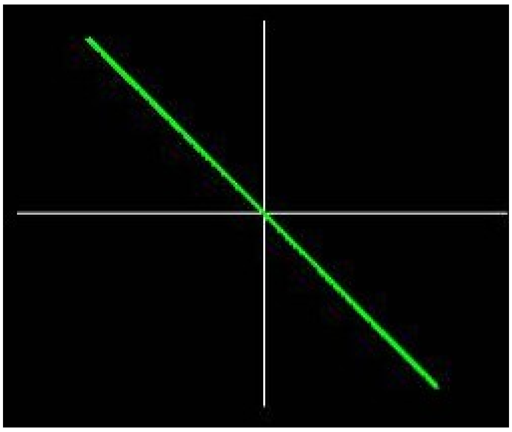
Lissajous Figures were first described in 1815 by Nathaniel Bowditch (1773-1838), who is best known today for his book, "The New American Practical Navigator", still available today. He also wrote widely on mathematics and astronomy, while pursuing a career as a navigator, surveyor, actuary and insurance company president, as well as being a member of the Corporation of Harvard College.

A Lissajous figure is produced by taking two sine waves and displaying them at right angles to each other. This is easily done on an oscilloscope in XY mode. Let's explain the phenomenon by taking two sine waves have equal amplitudes.

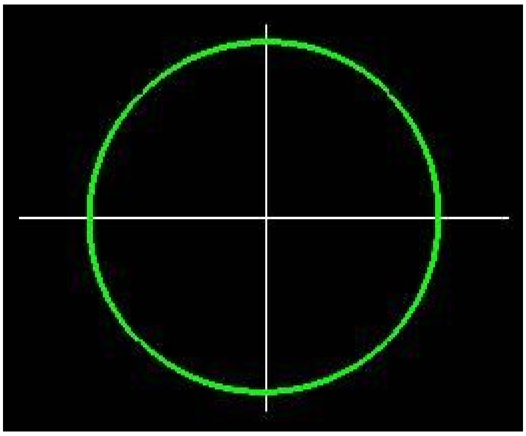
Case(I): When the two sine waves are of equal frequency and in-phase, we get a diagonal line to the right .



Case(II): When the two sine waves are of equal frequency and 180 degrees out-of-phase we get a diagonal line to the left.



Case(III): When the two sine waves are of equal frequency and 90 degrees out-of-phase we get a circle. This can be easily shown,

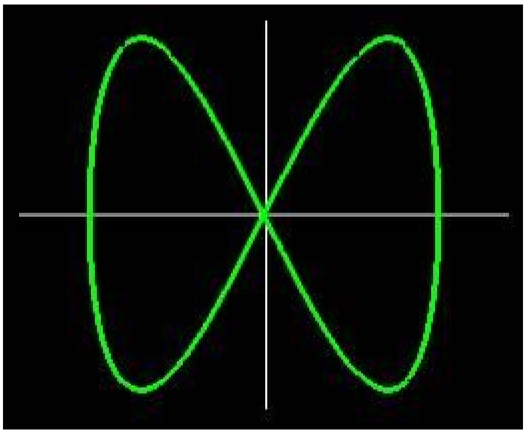


X = sin(a) and Y = sin(a + 90) = cos(a)

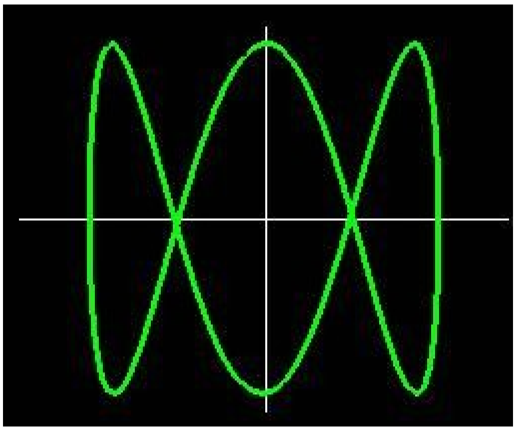
X*X + Y*Y = sin(a) * sin(a) + cos(a) * cos(a) = 1

which is the parametric equation for a circle having a radius of 1.

Case(IV): If the two sine waves are in phase but the frequency of the horizontal sine wave is twice the frequency of the vertical sine wave we get the pattern shown here.



Case(V): If the sine wave 90 degrees out-of-phase with the frequency of the horizontal sine wave three times the frequency of the vertical sine wave.



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Thanks

UNIT IV

Newtonian Relativity

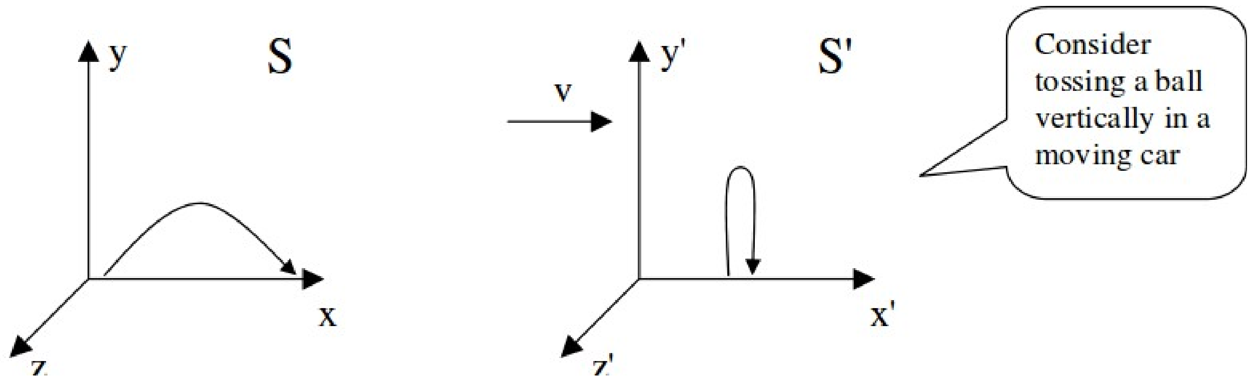
Galileo and Newton described the motion of objects with respect to a particular reference frame, which is basically a coordinate system attached to a particular observer.

A reference frame in which Newton's Laws hold is called an inertial frame. It is a frame that is not accelerating. Newtonian Principle of Relativity (Galilean Invariance):

If Newton's Laws hold in one inertial frame, they also hold in a reference frame moving at a constant velocity relative to the first frame. So the other frame is also an inertial frame. We can see this if we make a Galilean transformation:

Galilean Transformation

Consider a reference frame S' moving at a constant velocity with respect to a frame S:



x' = x+vt'   x = x' - vt'

y = y'   y' = y

z = z'   z' = z

t = t'   t' = t

These transformation equations show you how to convert a coordinate measured in one reference frame to the equivalent coordinate in the other reference frame. Implicit in a Galilean transformation is that time is universal (time runs at the same rate in all frames).

Now consider the action of a force in one reference frame. For example, the force of gravity causes a dropped ball to accelerate:

y component:

$$F'_{y} = ma'_{y} = m\frac{d^2y'}{dt^2}$$

But Since y' = y (t” = t)

$$a'_y = a_y \qquad \text{and} \qquad F'_y = F_y$$

x component:

$$F'_x = ma'_x = m\frac{d^2x'}{dt^2} = m\frac{d^2}{dt^2}(x - vt) = m\frac{d^2x}{dt^2} = F_x$$

$$a'_x = a_x \qquad \text{and} \qquad F'_x = F_x$$

Since the acceleration of the ball is the same in each reference frame, and thus the force acting on the ball, Newton's Laws are valid in both frames. Each is an inertial frame. Note that since the force is identical in each frame, there is noway to detect which frame is moving and which is not. You can only detect relative motion. For example, if a jet flies west at 1000 mph at the equator, is the jet moving or is the Earth moving?

The jet flies over the surface of the Earth, but with respect to the Sun the jet is not moving and the Earth is turning beneath it! The fact that we cannot detect absolute motion is known as Relativity. It is only relative motion that matters.

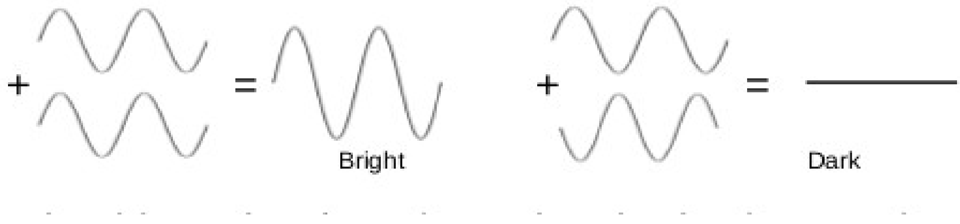
The Michelson-Morley Experiment

The Earth orbits around the sun at a high orbital speed, about 10^{-4}c, so an obvious experiment is to try to find the effects of the Earth's motion through the ether. Even though we don't know how fast the sun might be moving through the ether, the Earth's orbital velocity changes significantly throughout the year because of its change in direction, even if its orbital speed is nearly constant. Albert Michelson (1852– 1931) performed perhaps the most significant American physics experiment of the 1800s. Michelson, who was the first U.S. citizen to receive the Nobel Prize in Physics (1907), was an ingenious scientist who built an extremely precise device called an interferometer, which measures the phase difference between two light waves. Michelson used his interferometer to

detect the difference in the speed of light passing through the ether in different directions.

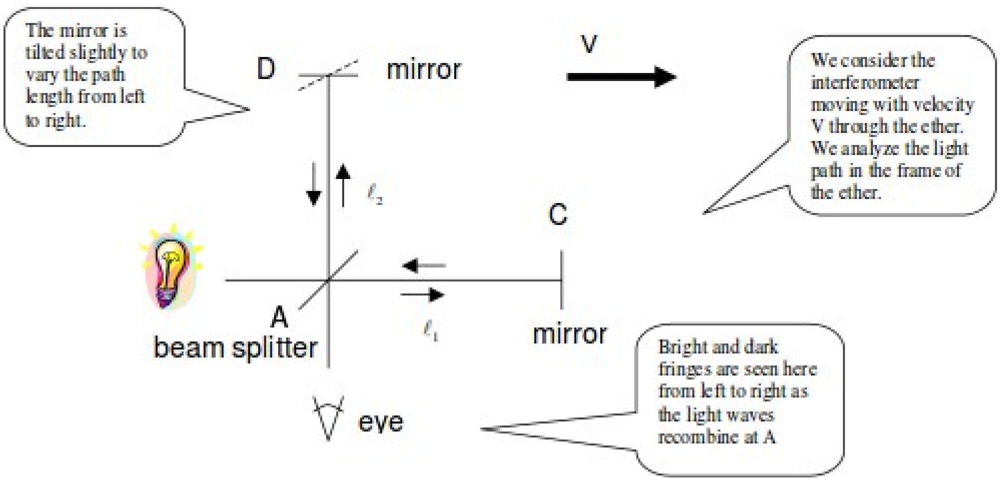
An interferometer was used to separate a light beam into two paths of possibly different length and then recombined. Since light is a wave, it exhibits the phenomenon of interference when multiple waves are combined. If two light waves are completely in phase, then the amplitude of each wave adds constructively . If they are completely out of phase, the amplitudes subtract destructively. Interferometers use monochromatic light so that the light wave consists of nearly a single wavelength. (Today we would use a laser).

Constructive interference Destructive interference



The basic technique is shown in Figure below:

The Michelson-Morely interferometer has two paths at right angles with respect to each other. It is at rest in a laboratory, presumably traveling through the ether. Considering that the velocity of light is c with respect to the ether, the distance light travels along each path is different even if the length of each "arm" is the same.

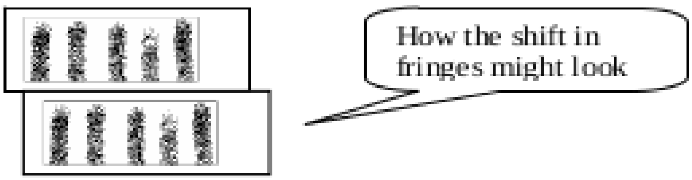


Although the experiment was sensitive enough to detect the expected ether drift, to every body's surprise nothing like that was found. The negative results gave two breath-taking results;

First, ether does not exist and so there is nothing like absolute motion relative to ether. All motion is relative to a specified frame of reference, not to a universal one

Second, the speed of light is same for all observers which is not true for waves requiring material medium for their propagation.

Conclusion: No shift was seen! Nor has one been seen ever since 1887. The conclusion must be that the ether does not exist. Light does not require any medium to propagation.



Special Theory of Relativity

The theory of relativity deals with the lack of Universal frame of reference. Special theory published by Einstein in 1905, treats problems that involves inertial frame of reference.

The theory can be explained under two postulates:

PI. The principle of relativity:

Laws of physics must be the same in all inertial reference frames.

Though this assertion may sound nothing new, it has to be appreciated that, first of all, this is a postulate. Besides, the change is in its privilege, now as an apriori assertion. The second postulate brings in some fundamental changes in our notion of space and time. While the following sections in this chapter are devoted to a more detailed discussion on these

aspects, we shall briefly define the bare minimum first, just enough material to state the postulate. The special theory forces us to look upon space and time not independently, but as a space-time continuum. Just as we speak of a point in 3-space given by 3-coordinates, we have Events(noun) designated by 4 coordinates - 3 spacial and 1 temporal. Thus we have a 4-dimensional space-time,and every space-time point is defined as an 'Event'. For a start it may be convenient to think of these Events as usual events(verb).

Consider two Events in space-time, say $(t_1;x_1;y_1;z_1)$,i = 1,2 (say two firecrackers bursting in the sky at two different points at different times). Contrary to our usual notion that time intervals

$\Delta t = (t_2 - t_1)$, and lengths $\Delta l^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$,are independently invariant in any reference frame, we have

PII. *The space time interval between two Events, defined as*

$$\Delta s^2 = c^2 \Delta t^2 - \Delta l^2,$$

is an invariant in any inertial reference frame, where ′c′

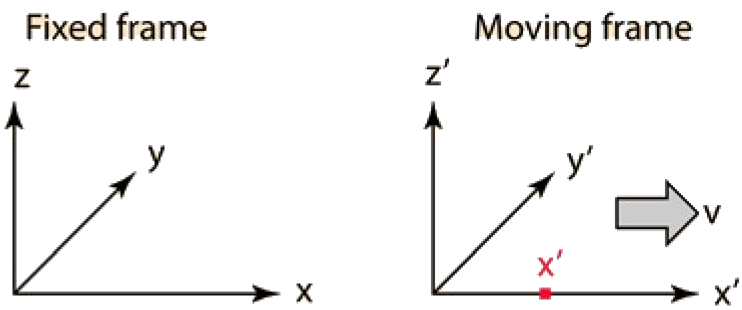
is a universal constant whose value is roughly 3×10^{8}m/s I.e., if the same two events are in another inertial reference frame designated by the space-time coordinates $(t'_i; x'_i, y'_i, z'_i)$, i= 1,2, then

$$c^2(t_2-t_1)^2-[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2]$$
$$=c^2(t'_2-t'_1)^2-[(x'_2-x'_1)^2+(y'_2-y'_1)^2+(z'_2-z'_1)^2]$$

The speed of light happens to be ′c′

Lorentz Transformations

As we switch from one reference frame to another, the simple velocity addition rule does not hold. Therefore we have to find the correct relativistic expression for adding velocities, the relations connecting space-time coordinates in two reference frames in relative motion. These relations are labeled as Lorentz transformation relations.



The primed frame moves with velocity v in the x direction with respect to the fixed reference frame. The reference frames coincide at t=t'=0. The point x' is moving with the primed frame.

The relations are given as;

$$\boxed{\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \end{aligned}}$$

The inverse relations are given as;

$$\boxed{x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

Here we can use;

$$\beta = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Therefore the modified form ofLorentz equations are;

$$\boxed{x' = \gamma\left(x - vt\right)}$$

$$\boxed{y' = y}$$

$$\boxed{z' = z}$$

$$\boxed{t' = \gamma\left(t - \frac{vx}{c^2}\right)}$$

Relativity of simultaneity

The relativity of simultaneity is the concept that *distant simultaneity* – whether two spatially separated events occur at the same time – is not absolute, but depends on the observer's reference frame . According to the special theory of relativity, it is impossible to say in an *absolute* sense that two distinct events occur at the same time if those events are separated in space. For example, a car crash in London and another in New York, which appear to happen at the same time to an observer on the earth, will appear to have occurred at slightly different times to an observer on an airplane flying between London and New York. The question of whether the events are simultaneous is *relative*: in the stationary earth reference frame the two accidents may happen at the same time but in other frames (in a different state of motion relative to the events) the crash in London may occur first, and in still other frames the New York crash may occur first. However, if the two events could be causally connected (i.e. the time between event A and event B is greater than the distance between them divided by the speed of light), the order is preserved (i.e., "event A precedes event B") in all frames of reference.

A mathematical form of the relativity of simultaneity ("local time") was introduced by H. Lorentz in 1892, and physically interpreted (to first order in v/c) as the result of a synchronization using light signals by Henri Poincare in 1900.

Length Contraction

Now let's consider what might happen to the length of objects in relativity. Let an observer in each system K and K' have a meter stick at rest in his or her own respective system. Each observer lays the stick down along his or her respective x axis, putting the left end at x_l(or x_l') and the right end at x_r (or x_r'). Thus, Frank in system K measures his stick to be $L_0 = x_r - x_l$. Similarly, in system K', Mary measures her stick at rest to be $L_0' = x_r' - x_l' = L_0$. Every observer measures a meter stick at rest in his or her own system to have the same length, namely one meter. The length as measured at rest is called the **proper length.**

Let system K be at rest and system K' move along the x axis with speed v. Frank, who is at rest in system K, measures the length of the stick moving in K'. The difficulty is to measure the ends of the stick simultaneously. We insist that Frank measure the ends of the stick at the same time so that $t=t_r= t_l$. The events denoted by (x, t) are (x_l, t) and (x_r , t). We use find;

$$x_r' - x_\ell' = \frac{(x_r - x_\ell) - v(t_r - t_\ell)}{\sqrt{1 - v^2/c^2}}$$

The meter stick is at rest in system K', so the length $x_r' - x_l'$ must be the proper length L_0'. Denote the length measured by Frank as $L = x_r - x_l$. The times t_r and t_l are identical, as we insisted, so $t_r - t_l = 0$. Notice that the times of measurement

by Mary in her system, t_r' and t_l', are not identical. It makes no difference when Mary makes the measurements in her own system, because the stick is at rest. However, it makes a big difference when Frank makes his measurements, because the stick is moving with speed v with respect to him. The measurements must be done simultaneously! With these results, the previous equation becomes

$$L_0' = \frac{L}{\sqrt{1 - v^2/c^2}} = \gamma L$$

or because $L_0 = L_0'$;

$$L = L_0\sqrt{1 - v^2/c^2} = \frac{L_0}{\gamma}$$

Notice that $L_0>L$, so the moving meter stick shrinks according to Frank. This effect is known as length or space contraction and is characteristic of relative motion.

Time Dilation

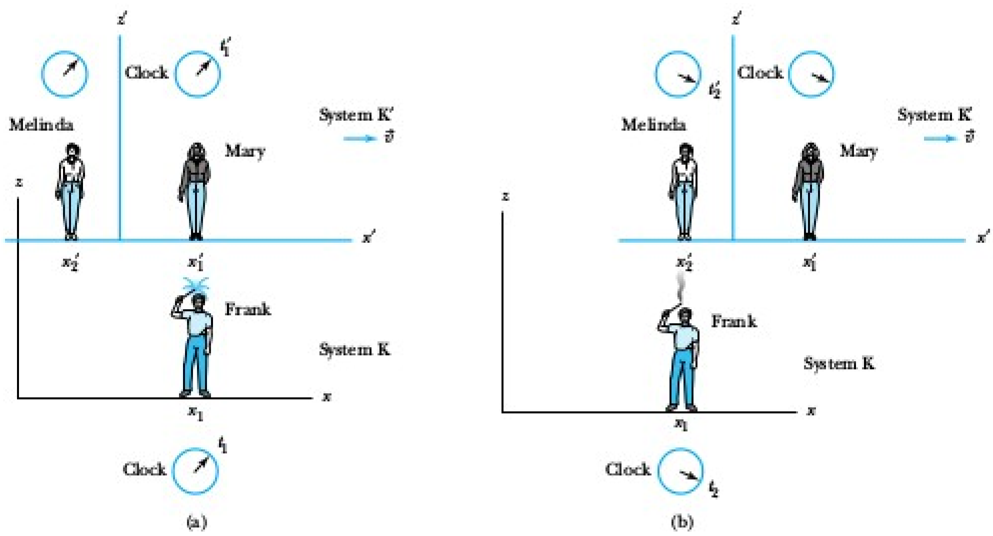
Consider again our two systems K and K' with system K fixed and system K' moving along the x axis with velocity v as shown in Figure below. Frank lights a sparkler at position x_1 in system K. A clock placed beside the sparkler indicates the time to be t_1 when the sparkler is lit and t_2 when the sparkler goes out . The sparkler burns for time T_0 , where $T_0 = t_2 - t_1$. The time difference between two events occurring at the same position in a system as measured by a clock at rest in the system is called the proper time. We use the subscript zero on the time difference T_0 to denote the proper time. Now what is the time as determined by Mary who is passing by (but at rest in her own system K')? All the clocks in both systems have been synchronized when the systems are at rest with respect to one another. The two events (sparkler lit and then going out) do not occur at the same place according to Mary.

She is beside the sparkler when it is lit, but she has moved far away from the sparkler when it goes out . Her friend Melinda, also at rest in system K', is beside the sparkler when it goes out. Mary and Melinda measure the two times for the sparkler to be lit and to go out in system K' as times t_1' andt_1' . The Lorentz transformation relates these times to those measured in system K as

$$t_2' - t_1' = \frac{(t_2 - t_1) - (v/c^2)(x_2 - x_1)}{\sqrt{1 - v^2/c^2}}$$

In system K the clock is fixed at x_1 , so $x_2 - x_1 = 0$; that is, the two events occur at the same position. The time t_2 - t_1 is the proper time T_0 , and we denote the time differencet_2' - t_1' as measured in the moving system K':

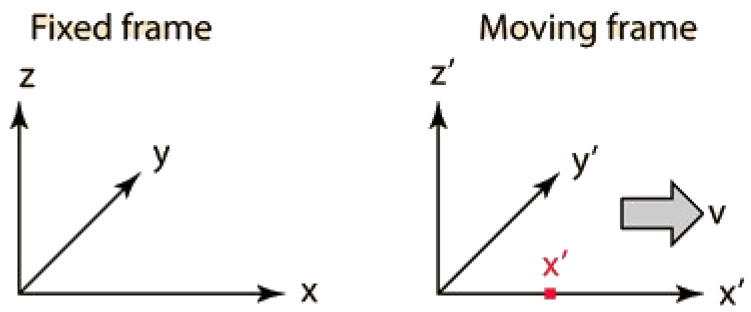
The pictorial representation of the different observation intervals are given below.



Thus, the Time Dilation is given as;

$$T' = \frac{T_0}{\sqrt{1 - v^2/c^2}} = \gamma T_0$$

Relativistic addition of velocities



If the primed frame is traveling with speed V in the positive x-direction relative to the unprimed frame then Lorentz transformations can be written as;

$$dx = \gamma_V (dx' + V dt'), \quad dy = dy', \quad dz = dz', \quad dt = \gamma_V \left(dt' + \frac{V}{c^2} dx' \right),$$

Where,

$$\gamma_V = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}.$$

Divide dx, dy, dz by dt we get;

$$\frac{dx}{dt} = \frac{\gamma_V\left(dx' + V dt'\right)}{\gamma_V\left(dt' + \frac{V}{c^2}dx'\right)}, \quad \frac{dy}{dt} = \frac{dy'}{\gamma_V\left(dt' + \frac{V}{c^2}dx'\right)}, \quad \frac{dz}{dt} = \frac{dz'}{\gamma_V\left(dt' + \frac{V}{c^2}dx'\right)},$$

Or,

$$\frac{dx}{dt} = \frac{dx' + V dt'}{dt'(1 + \frac{V}{c^2}\frac{dx'}{dt'})}, \quad \frac{dy}{dt} = \frac{dy'}{\gamma_V\ dt'(1 + \frac{V}{c^2}\frac{dx'}{dt'})}, \quad \frac{dz}{dt} = \frac{dz'}{\gamma_V\ dt'(1 + \frac{V}{c^2}\frac{dx'}{dt'})},$$

Thus in Cartesian coordinates the velocity transformation can be written as;

$$\boxed{v_x = \frac{v_x' + V}{1 + \frac{V}{c^2}v_x'}, \quad v_y = \frac{\sqrt{1 - \frac{V^2}{c^2}}\,v_y'}{1 + \frac{V}{c^2}v_x'}, \quad v_z = \frac{\sqrt{1 - \frac{V^2}{c^2}}\,v_z'}{1 + \frac{V}{c^2}v_x'}.}$$

Variation of Mass with Velocity

According to Newtonian mechanics the mass of a body does not change with velocity. However, conservation laws, especially here the law of conservation of momentum, hold for any inertial system. Hence, in order to maintain the momentum conserved in any isolated system, mass of the body must be related to its velocity. So according to Einstein, the mass of the body in motion is different from the mass of the body at rest. We consider two inertial frames S and S' as in Figure below;

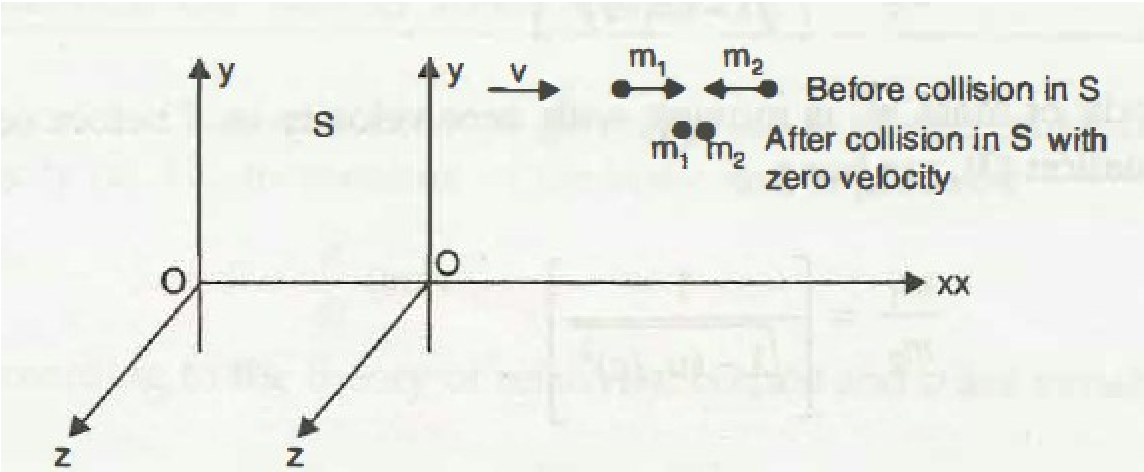


Fig: Collision between masses viewed from stationary and moving frames of reference.

We now consider the collision of two bodies in S' and view it from the S. Let the two particles of masses m_1 and m_2 are travelling with velocity u ' and-u ' parallel to x-axis in S'. The two bodies collide and after collision they coalesced into one body.

In System S : Before Collision: Mass of bodies are m1 and m2 •

Let the their velocities are u_1 and u_2 respectively.

In System S: After Collision: Mass of the coalesced body is $(m_1 + m_2)$ and the velocity Is v .Using law of addition of velocities;

$$u_1 = \frac{u' + v}{1 + \frac{u'v'}{c^2}} \quad \text{and} \quad u_2 = \frac{-u' + v}{1 - \frac{u'v}{c^2}}$$

Applying the principle of conservation of momentum of the system before and after the collision, we have,

$m_1 u_1 + m_2 u_2 = (m_1 + m_2)v$

$$m_1\left[\frac{u' + v}{1 + \frac{u'v}{c^2}}\right] + m_2\left[\frac{-u' + v}{1 - \frac{u'v}{c^2}}\right] = (m_1 + m_2)v$$

$$m_1\left[\frac{u' + v}{1 + \frac{u'v}{c^2}} - v\right] = m_2\left[v - \frac{-u' + v}{1 - \frac{u'v}{c^2}}\right]$$

$$\frac{m_1}{m_2} = \left[\frac{1 + \frac{u'v}{c^2}}{1 - \frac{u'v}{c^2}}\right]$$

Now, using equations (1) and (2), we have

$M_1/m_2 = [\surd\ 1-(u_2\ /c)^2\ /\surd\ 1-(u_1\ /c)^2\ ]$

Let the body of mass m_2 is moving with zero velocity in S before collision, i.e., $u_2 = 0$,hence, using equation (3), we have,

$$m_1 / m_2 = 1 / \sqrt{1-(u_1/c)^2}$$

Using common notation as $m_1 = m$, $m_2 = m_0$, $u_1 = v$, we have by using equation (4).

This is the relativistic formula for variation of mass with velocity, where m_0 is the rest mass and m is the relativistic mass of the body. There are a large numbers of experimental observations of this enhancement of mass of particles in high energy physics

I. When v << c

$v^2 << c^2$, v^2 / c^2 is negligible as compared to 1 => c $m = m_0$

When velocity of the moving particle is much smaller as compared to velocity of light, relativistic mass equals the rest mass.

II. When v= c

$V^2 = c^2$,$v^2 / c^2 = 1$ => $[1- v^2 / c^2]$,< 1 => $m > m_0$

When velocity of the moving particle is comparable to velocity of light, relativistic mass of the body appears to be greater than the rest mass.

III. When v = c

$V^2 = c^2$, $v^2 / c^2 = 1$ => m

When velocity of the moving particle is exactly equal to velocity of light, relativistic mass of the body appears to be infinite and this is an impractical concept.

IV. When v > c

$V^2 >> c^2, v^2/c^2 > 0$ m = Imaginary

When velocity of the moving particle is greater as compared to velocity of light., relativistic mass becomes imaginary and this is an impractical concept.

Mass Energy Relation

The $E=mc^2$ relationship between mass and energy was first made explicit in a short piece by Einstein ("On the Origin of Inertia") which was written as a postscript to the famous 1905 "Electrodynamics" paper, and which presented the $E=mc^2$ result as a consequence of the mathematical relationships that had appeared in the earlier piece. W. L. Fadner has also unearthed and discussed a number of contemporary pieces that either came close to deriving $E=mc^2$, or presented similar equations without fully exploring the consequences or claiming the result to be general.

Consider an object of rest mass m'. If force is applied to the object such that it starts moving with relativistic velocity (that is comparable with the speed of light), then its mass will also vary with variation of mass with energy relation

$$m = m'/(1-v^2/c^2)^{1/2} \qquad (1)$$

Now suppose that work dw will be done due to this force. If the object is displaced along x axis, then work will be:

$dw = Fdx$

or $dw = (dp/dt)dx$ (because from Newton's 2nd law $F = dp/dt$)

or $dw = [d(mv)/dt]dx$ (because $p = mv$)

Differentiate R.H.S.

$dw = (mdv/dt + vdm/dt)dx$ (here m is also a variable quantity, thus m is also differentiated)

or $dw = mdvdx/dt + vdmdx/dt$

$$\text{or } dw = mvdv + v^2dm \qquad (2)$$

Now square equation (1) and cross-multiply

$m^2 (1-v^2/c^2) = m'^2$

or $m^2 [(c^2-v^2)/c^2] = m'^2$

or $m^2c^2 - m^2v^2 = m'^2c^2$

Differentiating, we get

$$c^2(2mdm) – m^2(2vdv) – v^2(2mdm)$$

or $v^2dm + mvdv = c^2dm$ (3)

Comparing equations (2) and (3), we get

$dw = c^2dm$ (4)

The total amount of work done by the applied force in order to change its velocity from 0 to v (or mass from m' to m) is achieved by integrating the L.H.S of the following equation with limits 0 to W and R.H.S. from m' to m (because when work is 0 then body has rest mass m' and when work W is done then body has variable mass m).

$$\int dw = c^2 \int dm$$

Or $W = c^2(m – m')$ (5)

As this work W is done to give motion to the object. Therefore, W will appear in the form of kinetic energy acquired by the body, Thus relativistic kinetic energy will be

$K = = c^2(m – m')$ (6)

By definition of potential energy or the rest mass energy, it is equal to the internal energy of the body. It is also equal to the work done to bring all the particles which make the object of rest mass m'. Thus the rest mass energy of the body is derived as by integrating the L.H.S of the following equation with limits 0 to W and R.H.S. from 0 to m (because when work is 0 then body has rest mass does not exist and when work W is done then all the particles make an object of rest mass m').

$\int dw = c^2 \int dm$

Thus $W = m'c^2$

Therefore, W will appear in the form of rest mass energy of the body, Thus rest mass energy will be

$$R = m'c^2 \qquad (7)$$

The total energy of the object will be

E = kinetic energy + rest mass energy

Put equations (6) and (7) in this equation, we get

$E = c^2(m - m') + m'c^2$

Or $E = mc^2$

This is the famous Einstein mass-energy equivalence relation.

Space-time four-dimensional continuum

A four-dimensional reference frame, consisting of three dimensions in space and one dimension in time, used especially in Relativity Theory as a basis for coordinate systems for identifying the location and timing of objects and events. In **General Relativity**, space-time is thought to be curved by the presence of mass, much as the space defined by the surface of a piece of paper can be curved by bending the paper.

The general expression is written as;

$$s^2 = x^2 + y^2 + z^2 - (ct)^2$$

For simplicity, we will sometimes use only the single spatial coordinate x. If we consider two events, we can determine the quantity Δs^2 where;

$$\Delta s^2 = \Delta x^2 - c^2 \Delta t^2$$

between the two events, and we find that it is invariant in any inertial frame. The quantity Δs is known as the space-time interval between two events. There are three possibilities for the invariant quantity Δs^2.

1. $\Delta s^2 = 0$: In this case $\Delta x^2 = c^2 \Delta t^2$, and the two events can be connected only by a light signal. The events are said to have a light-like separation.

2. $\Delta s^2 > 0$: Here we must have $\Delta x^2 > c^2 \Delta t^2$, and no signal can travel fast enough to connect the two events. The events are not causally connected and are said to have a space-like separation. In this case we can always find an inertial frame traveling at a velocity less than c in which the two events can occur simultaneously in time but at different places in space.

3. $\Delta s^2 < 0$: Here we have $\Delta x^2 < c^2 \Delta t^2$, and the two events can be causally connected. The interval is said to be time-like. In this case we can find an inertial frame traveling at a velocity less than c in which the two events occur at the same position in space but at different times. The two events can never occur simultaneously.

Four-vectors

In special Relativity, a four-vector (also known as a 4-vector) is an object with four components, which transform in a specific way under Lorentz Transformations. Specifically, a four-vector is an element of a four-dimensional vector space considered as a representation space of the representation of the Lorentz group.

In the literature of relativity, space-time coordinates and the energy/momentum of a particle are often expressed in four-vector form. They are defined so that the length of a four-vector is invariant under a coordinate transformation. This invariance is associated with physical ideas. The invariance of the space-time four-vector is associated with the fact that the speed of light is a constant. The invariance of the energy-momentum four-vector is associated with the fact that the rest mass of a particle is invariant under coordinate transformations.

The space-time 4-vector is defined by

$$\boxed{\vec{R} = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ct \\ \vec{r} \end{bmatrix}}$$

The energy-momentum 4-vector is defined by

$$\boxed{\vec{P} = \begin{bmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{bmatrix} = \begin{bmatrix} E \\ \vec{p}c \end{bmatrix}}$$

The scalar product of two space-time 4-vectors is defined by

$$\boxed{\vec{R}_a = \begin{bmatrix} ct \\ \vec{r}_a \end{bmatrix} \quad \vec{R}_b = \begin{bmatrix} ct \\ \vec{r}_b \end{bmatrix} \quad \vec{R}_a \cdot \vec{R}_b = ct_a ct_b - \vec{r}_a \cdot \vec{r}_b}$$

Note that this differs from the ordinary scalar product of vectors because of the minus sign. That minus sign is necessary for the property of invariance of the length of the 4-vectors.

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Thanks