1. INTRODUCTION TO OR

1.1 TERMINOLOGY

The British/Europeans refer to "operational research", the Americans to "operations research" - but both are often shortened to just "OR" (which is the term we will use). Another term which is used for this field is "management science" ("MS"). The Americans sometimes combine the terms OR and MS together and say "***OR/MS***" or "ORMS".

Yet other terms sometimes used are "industrial engineering" ("IE"), "decision science" ("DS"), and “problem solving”.

In recent years there has been a move towards a standardization upon a single term for the field, namely the term "OR".

“Operations Research (Management Science) is a scientific approach to decision making that seeks to best design and operate a system, usually under conditions requiring the allocation of scarce resources.”

A system is an organization of interdependent components that work together to accomplish the goal of the system.

1.2 THE METHODOLOGY OF OR

When OR is used to solve a problem of an organization, the following seven step procedure should be followed:

<u>Step 1. Formulate the Problem</u>

OR analyst first defines the organization's problem. Defining the problem includes specifying the organization's objectives and the parts of the organization (or system) that must be studied before the problem can be solved.

<u>Step 2. Observe the System</u>

Next, the analyst collects data to estimate the values of parameters that affect the organization's problem. These estimates are used to develop (in Step 3) and evaluate (in Step 4) a mathematical model of the organization's problem.

Step 3. Formulate a Mathematical Model of the Problem

The analyst, then, develops a mathematical model (in other words an idealized representation) of the problem. In this class, we describe many mathematical techniques that can be used to model systems.

Step 4. Verify the Model and Use the Model for Prediction

The analyst now tries to determine if the mathematical model developed in Step 3 is an accurate representation of reality. To determine how well the model fits reality, one determines how valid the model is for the current situation.

Step 5. Select a Suitable Alternative

Given a model and a set of alternatives, the analyst chooses the alternative (if there is one) that best meets the organization's objectives.

Sometimes the set of alternatives is subject to certain restrictions and constraints. In many situations, the best alternative may be impossible or too costly to determine.

Step 6. Present the Results and Conclusions of the Study

In this step, the analyst presents the model and the recommendations from Step 5 to the decision making individual or group. In some situations, one might present several alternatives and let the organization choose the decision maker(s) choose the one that best meets her/his/their needs.

After presenting the results of the OR study to the decision maker(s), the analyst may find that s/he does not (or they do not) approve of the recommendations. This may result from incorrect definition of the problem on hand or from failure to involve decision maker(s) from the start of the project. In this case, the analyst should return to Step 1, 2, or 3.

Step 7. Implement and Evaluate Recommendation

If the decision maker(s) has accepted the study, the analyst aids in implementing the recommendations. The system must be constantly monitored (and updated dynamically as the environment changes) to ensure that the recommendations are enabling decision maker(s) to meet her/his/their objectives.

1.3 HISTORY OF OR

(Prof. Beasley’s lecture notes)

OR is a relatively new discipline. Whereas 70 years ago it would have been possible to study mathematics, physics or engineering (for example) at university it would not have been possible to study OR, indeed the term OR did not exist then. It was only really in the late 1930's that operational research began in a systematic fashion, and it started in the UK.

Early in 1936 the British Air Ministry established Bawdsey Research Station, on the east coast, near Felixstowe, Suffolk, as the centre where all pre-war radar experiments for both the Air Force and the Army would be carried out. Experimental radar equipment was brought up to a high state of reliability and ranges of over 100 miles on aircraft were obtained.

It was also in 1936 that Royal Air Force (RAF) Fighter Command, charged specifically with the air defense of Britain, was first created. It lacked however any effective fighter aircraft - no Hurricanes or Spitfires had come into service - and no radar data was yet fed into its very elementary warning and control system.

It had become clear that radar would create a whole new series of problems in fighter direction and control so in late 1936 some experiments started at Biggin Hill in Kent into the effective use of such data. This early work, attempting to integrate radar data with ground based observer data for fighter interception, was the start of OR.

The first of three major pre-war air-defense exercises was carried out in the summer of 1937. The experimental radar station at Bawdsey Research Station was brought into operation and the information derived from it was fed into the general air-defense warning and control system. From the early warning point of view this exercise was encouraging, but the tracking information obtained from radar, after filtering and transmission through the control and display network, was not very satisfactory.

In July 1938 a second major air-defense exercise was carried out. Four additional radar stations had been installed along the coast and it was hoped that Britain now had an aircraft location and control system greatly improved both in coverage and effectiveness. Not so! The exercise revealed, rather, that a new and serious problem had arisen. This was the need to coordinate and correlate the additional, and often conflicting, information received from the additional radar stations. With the out-break of war apparently imminent, it was obvious that something new - drastic if necessary - had to be attempted. Some new approach was needed.

Accordingly, on the termination of the exercise, the Superintendent of Bawdsey Research Station, A.P. Rowe, announced that although the exercise had again demonstrated the technical feasibility of the radar system for detecting aircraft, its operational achievements still fell far short of requirements. He therefore proposed that a crash program of research into the operational - as opposed to the technical -

aspects of the system should begin immediately. The term "operational research" [RESEARCH into (military) OPERATIONS] was coined as a suitable description of this new branch of applied science. The first team was selected from amongst the scientists of the radar research group the same day.

In the summer of 1939 Britain held what was to be its last pre-war air defense exercise. It involved some 33,000 men, 1,300 aircraft, 110 antiaircraft guns, 700 searchlights, and 100 barrage balloons. This exercise showed a great improvement in the operation of the air defense warning and control system. The contribution made by the OR teams was so apparent that the Air Officer Commander-in-Chief RAF Fighter Command (Air Chief Marshal Sir Hugh Dowding) requested that, on the outbreak of war, they should be attached to his headquarters at Stanmore.

On May 15th 1940, with German forces advancing rapidly in France, Stanmore Research Section was asked to analyze a French request for ten additional fighter squadrons (12 aircraft a squadron) when losses were running at some three squadrons every two days. They prepared graphs for Winston Churchill (the British Prime Minister of the time), based upon a study of current daily losses and replacement rates, indicating how rapidly such a move would deplete fighter strength. No aircraft were sent and most of those currently in France were recalled.

This is held by some to be the most strategic contribution to the course of the war made by OR (as the aircraft and pilots saved were consequently available for the successful air defense of Britain, the Battle of Britain).

In 1941 an Operational Research Section (ORS) was established in Coastal Command which was to carry out some of the most well-known OR work in World War II.

Although scientists had (plainly) been involved in the hardware side of warfare (designing better planes, bombs, tanks, etc) scientific analysis of the operational use of military resources had never taken place in a systematic fashion before the Second World War. Military personnel, often by no means stupid, were simply not trained to undertake such analysis.

These early OR workers came from many different disciplines, one group consisted of a physicist, two physiologists, two mathematical physicists and a surveyor. What such people brought to their work were "scientifically trained" minds, used to querying assumptions, logic, exploring hypotheses, devising experiments, collecting data, analyzing numbers, etc. Many too were of high intellectual caliber (at least four wartime OR personnel were later to win Nobel prizes when they returned to their peacetime disciplines).

By the end of the war OR was well established in the armed services both in the UK and in the USA.

OR started just before World War II in Britain with the establishment of teams of scientists to study the strategic and tactical problems involved in military operations. The objective was to find the most effective utilization of limited military resources by the use of quantitative techniques.

Following the end of the war OR spread, although it spread in different ways in the UK and USA.

You should be clear that the growth of OR since it began (and especially in the last 30 years) is, to a large extent, the result of the increasing power and widespread availability of computers. Most (though not all) OR involves carrying out a large number of numeric calculations. Without computers this would simply not be possible.

2. BASIC OR CONCEPTS

"OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing."

We can also define a mathematical model as consisting of:

- *Decision variables*, which are the unknowns to be determined by the solution to the model.
- *Constraints* to represent the physical limitations of the system
- An *objective* function
- An *optimal solution* to the model is the identification of a set of variable values which are feasible (satisfy all the constraints) and which lead to the optimal value of the objective function.

An optimization model seeks to find values of the decision variables that optimize (maximize or minimize) an objective function among the set of all values for the decision variables that satisfy the given constraints.

Two Mines Example

The Two Mines Company own two different mines that produce an ore which, after being crushed, is graded into three classes: high, medium and low-grade. The company has contracted to provide a smelting plant with 12 tons of high-grade, 8 tons of medium-grade and 24 tons of low-grade ore per week. The two mines have different operating characteristics as detailed below.

Mine	Cost per day (£'000)	Production (tons/day)		
		High	Medium	Low
X	180	6	3	4
Y	160	1	1	6

Consider that mines cannot be operated in the weekend. How many days per week should each mine be operated to fulfill the smelting plant contract?

Guessing

To explore the Two Mines problem further we might simply guess (i.e. use our judgment) how many days per week to work and see how they turn out.

work one day a week on *X*, one day a week on *Y*

This does not seem like a good guess as it results in only 7 tones a day of high-grade, insufficient to meet the contract requirement for 12 tones of high-grade a day. We say that such a solution is *infeasible*.

work 4 days a week on *X*, 3 days a week on *Y*

This seems like a better guess as it results in sufficient ore to meet the contract. We say that such a solution is *feasible*. However it is quite expensive (costly).

We would like a solution which supplies what is necessary under the contract at minimum cost. Logically such a minimum cost solution to this decision problem must exist. However even if we keep guessing we can never be sure whether we have found this minimum cost solution or not. Fortunately our structured approach will enable us to find the minimum cost solution.

Solution

What we have is a verbal description of the Two Mines problem. What we need to do is to translate that verbal description into an *equivalent* mathematical description.

In dealing with problems of this kind we often do best to consider them in the order:

Variables

Constraints

Objective

This process is often called *formulating* the problem (or more strictly formulating a mathematical representation of the problem).

<u>Variables</u>

These represent the "decisions that have to be made" or the "unknowns". We have two decision variables in this problem:

x = number of days per week mine *X* is operated

y = number of days per week mine *Y* is operated

Note here that $x \geq 0$ and $y \geq 0$.

Constraints

It is best to first put each constraint into words and then express it in a mathematical form.

ore production constraints - balance the amount produced with the quantity required under the smelting plant contract

Ore	
High	$6x + 1y \geq 12$
Medium	$3x + 1y \geq 8$
Low	$4x + 6y \geq 24$

days per week constraint - we cannot work more than a certain maximum number of days a week e.g. for a 5 day week we have

$x \leq 5$

$y \leq 5$

Inequality constraints

Note we have an inequality here rather than an equality. This implies that we may produce more of some grade of ore than we need. In fact we have the general rule: given a choice between an equality and an inequality choose the inequality

For example - if we choose an equality for the ore production constraints we have the three equations $6x+y=12$, $3x+y=8$ and $4x+6y=24$ and there are no values of x and y which satisfy all three equations (the problem is therefore said to be "over-constrained"). For example the values of x and y which satisfy $6x+y=12$ and $3x+y=8$ are $x=4/3$ and $y=4$, but these values do not satisfy $4x+6y=24$.

The reason for this general rule is that choosing an inequality rather than an equality gives us more flexibility in optimizing (maximizing or minimizing) the objective (deciding values for the decision variables that optimize the objective).

Implicit constraints

Constraints such as days per week constraint are often called implicit constraints because they are implicit in the definition of the variables.

Objective

Again in words our objective is (presumably) to minimize cost which is given by $180x + 160y$

Hence we have the ***complete mathematical representation*** of the problem:

Minimize

$180x + 160y$

subject to

$6x + y \geq 12$

$3x + y \geq 8$

$4x + 6y \geq$

$24\ x \leq 5$

$y \leq 5\ x,$

$y \geq 0$

Some notes

The mathematical problem given above has the form

all variables continuous (i.e. can take fractional values)

a single objective (maximize or minimize)

the objective and constraints are linear i.e. any term is either a constant or a constant multiplied by an unknown (e.g. 24, $4x$, $6y$ are linear terms but xy or x^2 is a non-linear term)

Any formulation which satisfies these three conditions is called a *linear program* (LP). We have (implicitly) assumed that it is permissible to work in fractions of days - problems where this is not permissible and variables must take integer values will be dealt with under *Integer Programming* (IP).

Discussion

This problem was a *decision problem.*

We have taken a real-world situation and constructed an equivalent mathematical representation - such a representation is often called a mathematical *model* of the real-world situation (and the process by which the model is obtained is called *formulating* the model).

Just to confuse things the mathematical model of the problem is sometimes called the *formulation* of the problem.

Having obtained our mathematical model we (hopefully) have some quantitative method which will enable us to numerically solve the model (i.e. obtain a numerical solution) - such a quantitative method is often called an *algorithm* for solving the model.

Essentially an algorithm (for a particular model) is a set of instructions which, when followed in a step-by-step fashion, will produce a numerical solution to that model.

Our model has an *objective*, that is something which we are trying to *optimize*. Having obtained the numerical solution of our model we have to translate that solution back into the real-world situation.

"OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing."

3. LINEAR PROGRAMMING

It can be recalled from the Two Mines example that the conditions for a mathematical model to be a linear program (LP) were:

all variables continuous (i.e. can take fractional values)
a single objective (minimize or maximize)

the objective and constraints are linear i.e. any term is either a constant or a constant multiplied by an unknown.

LP's are important - this is because:

many practical problems can be formulated as LP's

there exists an algorithm (called the *simplex* algorithm) which enables us to solve LP's numerically relatively easily

We will return later to the simplex algorithm for solving LP's but for the moment we will concentrate upon formulating LP's.

Some of the major application areas to which LP can be applied are:

Work scheduling

Production planning & Production process Capital budgeting

Financial planning

Blending (e.g. Oil refinery management)
Farm planning

Distribution

Multi-period decision problems

- Inventory model
- Financial models
- Work scheduling

Note that the key to formulating LP's is practice. However a useful hint is that common objectives for LP's are maximize profit/minimize cost.

There are four basic assumptions in LP:

Proportionality

- The contribution to the objective function from each decision variable is proportional to the value of the decision variable (The contribution to the objective function from making four soldiers (4 $3=$12) is exactly four times the contribution to the objective function from making one soldier ($3))
- The contribution of each decision variable to the LHS of each constraint is proportional to the value of the decision variable (It takes exactly three times as many finishing hours (2hrs 3=6hrs) to manufacture three soldiers as it takes to manufacture one soldier (2 hrs))

Additivity

- The contribution to the objective function for any decision variable is independent of the values of the other decision variables (No matter what the value of train (x_2), the manufacture of soldier (x_1) will always contribute $3x_1$ dollars to the objective function)
- The contribution of a decision variable to LHS of each constraint is independent of the values of other decision variables (No matter what the value of x_1, the manufacture of x_2 uses x_2 finishing hours and x2 carpentry hours)
  - *1st implication*: The value of objective function is the sum of the contributions from each decision variables.
  - *2nd implication*: LHS of each constraint is the sum of the contributions from each decision variables.

Divisibility

- Each decision variable is allowed to assume fractional values. If we actually can not produce a fractional number of decision variables, we use IP (It is acceptable to produce 1.69 trains)

Certainty

- Each parameter is known with certainty

3.1 FORMULATING LP

3.1.1 Giapetto Example

Giapetto's wooden soldiers and trains. Each soldier sells for \$27, uses \$10 of raw materials and takes \$14 of labor & overhead costs. Each train sells for \$21, uses \$9 of raw materials, and takes \$10 of overhead costs. Each soldier needs 2 hours finishing and 1 hour carpentry; each train needs 1 hour finishing and 1 hour carpentry. Raw materials are unlimited, but only 100 hours of finishing and 80 hours of carpentry are available each week. Demand for trains is unlimited; but at most 40 soldiers can be sold each week. How many of each toy should be made each week to maximize profits?

Answer

Decision variables completely describe the decisions to be made (in this case, by Giapetto). Giapetto must decide how many soldiers and trains should be manufactured each week. With this in mind, we define:

x_1 = the number of soldiers produced per week

x_2 = the number of trains produced per week

Objective function is the function of the decision variables that the decision maker wants to maximize (revenue or profit) or minimize (costs). Giapetto can concentrate on maximizing the total weekly profit (z).

Here profit equals to (weekly revenues) – (raw material purchase cost) – (other variable costs). Hence Giapetto's objective function is:

Maximize $z = 3x_1 + 2x_2$

Constraints show the restrictions on the values of the decision variables. Without constraints Giapetto could make a large profit by choosing decision variables to be very large. Here there are three constraints:

Finishing time per week

Carpentry time per week

Weekly demand for soldiers

Sign restrictions are added if the decision variables can only assume nonnegative values (Giapetto cannot manufacture negative number of soldiers or trains!)

All these characteristics explored above give the following ***Linear Programming*** (LP) model

$$\begin{array}{lll} \max z = 3x_1 + 2x_2 & & \text{(The Objective function)} \\ \text{s.t.} & 2x_1 + x_2 \leq 100 & \text{(Finishing constraint)} \\ & x_1 + x_2 \leq 80 & \text{(Carpentry constraint)} \\ & x_1 \geq 40 & \text{(Constraint on demand for soldiers)} \\ & x_1, x_2 \geq 0 & \text{(Sign restrictions)} \end{array}$$

A value of (x_1, x_2) is in the ***feasible region*** if it satisfies all the constraints and sign restrictions.

Graphically and computationally we see the solution is (x_1, x_2) = (20, 60) at which z = 180. (***Optimal solution***)

Report

The maximum profit is $180 by making 20 soldiers and 60 trains each week. Profit is limited by the carpentry and finishing labor available. Profit could be increased by buying more labor.

3.1.2 Advertisement Example

Dorian makes luxury cars and jeeps for high-income men and women. It wishes to advertise with 1 minute spots in comedy shows and football games. Each comedy spot costs $50K and is seen by 7M high-income women and 2M high-income men. Each football spot costs $100K and is seen by 2M high-income women and 12M high-income men. How can Dorian reach 28M high-income women and 24M high-income men at the least cost?

Answer

The decision variables are

x_1 = the number of comedy spots
x_2 = the number of football spots

The model of the problem:

$$\begin{array}{ll} \min z = 50x_1 + 100x_2 & \\ \text{st} & 7x_1 + 2x_2 \leq 28 \\ & 2x_1 + 12x_2 \leq 24 \\ & x_1, x_2 \;\; 0 \end{array}$$

The graphical solution is z = 320 when (x_1, x_2) = (3.6, 1.4). From the graph, in this problem rounding up to (x_1, x_2) = (4, 2) gives the best *integer* solution.

Report

The minimum cost of reaching the target audience is $400K, with 4 comedy spots and 2 football slots. The model is dubious as it does not allow for saturation after repeated viewings.

3.1.3 Diet Example

Ms. Fidan's diet requires that all the food she eats come from one of the four "basic food groups". At present, the following four foods are available for consumption: brownies, chocolate ice cream, cola, and pineapple cheesecake. Each brownie costs 0.5$, each scoop of chocolate ice cream costs 0.2$, each bottle of cola costs 0.3$, and each pineapple cheesecake costs 0.8$. Each day, she must ingest at least 500 calories, 6 oz of chocolate, 10 oz of sugar, and 8 oz of fat. The nutritional content per unit of each food is shown in Table. Formulate an LP model that can be used to satisfy her daily nutritional requirements at minimum cost.

	Calories	Chocolate (ounces)	Sugar (ounces)	Fat (ounces)
Brownie	400	3	2	2
Choc. ice cream (1 scoop)	200	2	2	4
Cola (1 bottle)	150	0	4	1
Pineapple cheesecake (1 piece)	500	0	4	5

Answer

The decision variables:

x_1: number of brownies eaten daily

x_2: number of scoops of chocolate ice cream eaten daily

x_3: bottles of cola drunk daily

x_4: pieces of pineapple cheesecake eaten daily

The objective function (the total cost of the diet in cents):

$$\min w = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

Constraints:

$$400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500 \quad \text{(daily calorie intake)}$$

$$3x_1 + 2x_2 \geq 6 \quad \text{(daily chocolate intake)}$$

$$2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10 \quad \text{(daily sugar intake)}$$

$$2x_1 + 4x_2 + x_3 + 5x_4 \geq 8 \quad \text{(daily fat intake)}$$

$$x_i \geq 0, \; i = 1, 2, 3, 4 \quad \text{(Sign restrictions!)}$$

Report

The minimum cost diet incurs a daily cost of 90 cents by eating 3 scoops of chocolate and drinking 1 bottle of cola ($w = 90$, $x_2 = 3$, $x_3 = 1$)

3.2 SOLVING LP

3.2.1 LP Solutions: Four Cases

When an LP is solved, one of the following four cases will occur:

1. The LP has a **unique optimal solution**.
2. The LP has **alternative (multiple) optimal solutions**. It has more than one (actually an infinite number of) optimal solutions
3. The LP is **infeasible**. It has no feasible solutions (The feasible region contains no points).
4. The LP is **unbounded**. In the feasible region there are points with arbitrarily large (in a max problem) objective function values.

3.2.2 The Graphical Solution

Any LP with only two variables can be solved graphically

Example 1. Giapetto

Since the Giapetto LP has two variables, it may be solved graphically.

Answer

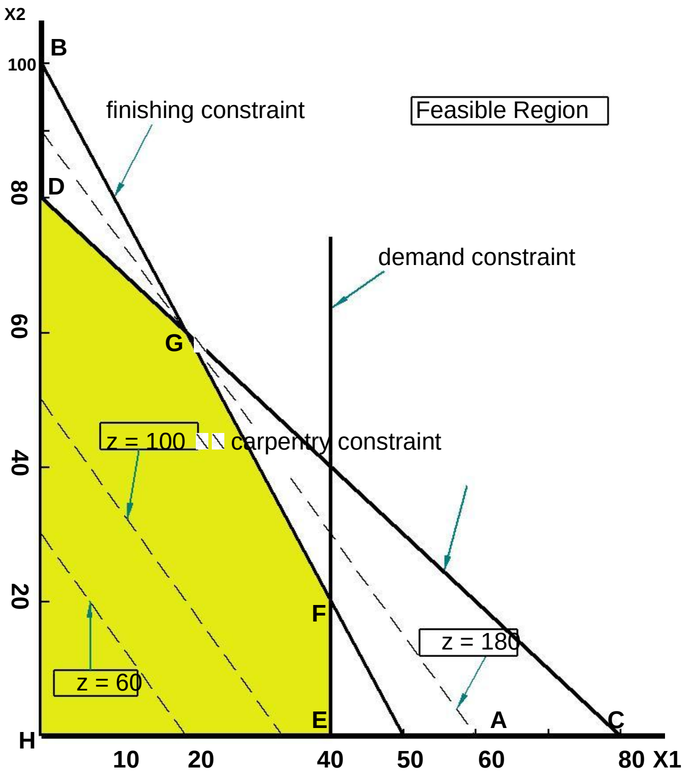
The feasible region is the set of all points satisfying the constraints.

$$\max z = 3x_1 + 2x_2$$

$$\begin{aligned}
\text{s.t.} \quad & 2x_1 + x_2 \leq 100 && \text{(Finishing constraint)} \\
& x_1 + x_2 \leq 80 && \text{(Carpentry constraint)} \\
& x_1 \leq 40 && \text{(Demand constraint)} \\
& x_1, x_2 \geq 0 && \text{(Sign restrictions)}
\end{aligned}$$

The set of points satisfying the LP is bounded by the five sided polygon DGFEH. Any point ***on*** or ***in*** the interior of this polygon (the shade area) is in the ***feasible region***. Having identified the feasible region for the LP, a search can begin for the ***optimal solution*** which will be the point in the feasible region with the *largest* z-value (maximization problem).

To find the optimal solution, a line on which the points have the same z-value is graphed. In a max problem, such a line is called an ***isoprofit*** line while in a min problem, this is called the ***isocost*** line. (*The figure shows the isoprofit lines for* $z = 60$, $z = 100$, *and* $z = 180$).



In the unique optimal solution case, isoprofit line last hits a point (vertex - corner) before leaving the feasible region.

The optimal solution of this LP is point G where (x_1, x_2) = (20, 60) giving z = 180.

A constraint is ***binding*** (active, tight) if the left-hand and right-hand side of the constraint are equal when the optimal values of the decision variables are substituted into the constraint.

A constraint is ***nonbinding*** (inactive) if the left-hand side and the right-hand side of the constraint are unequal when the optimal values of the decision variables are substituted into the constraint.

In Giapetto LP, the finishing and carpentry constraints are binding. On the other hand the demand constraint for wooden soldiers is nonbinding since at the optimal solution $x_1 < 40$ ($x_1 = 20$).

Example 2. Advertisement

(*Winston 3.2, p. 61*)

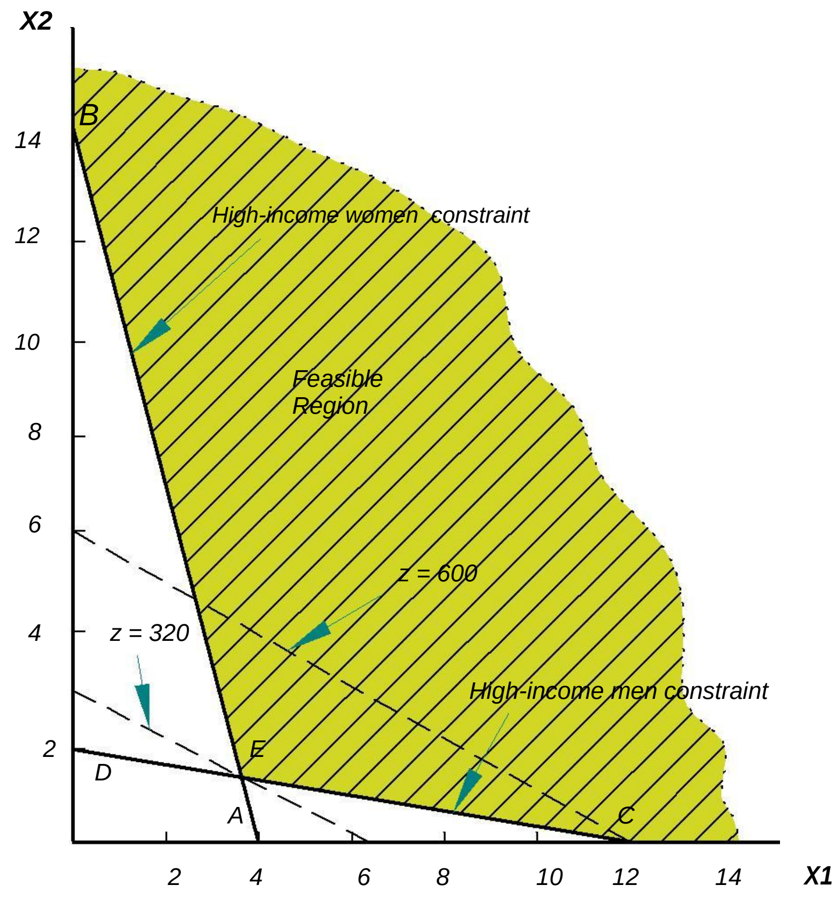
Since the Advertisement LP has two variables, it may be solved graphically.

Answer

The feasible region is the set of all points satisfying the constraints.

$$\min z = 50x_1 + 100x_2$$

$$\begin{aligned} \text{s.t.} \quad & 7x_1 + 2x_2 \geq 28 && \text{(high income women)} \\ & 2x_1 + 12x_2 \geq 24 && \text{(high income men)} \\ & x_1, x_2 \geq 0 \end{aligned}$$



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Since Dorian wants to minimize total advertising costs, the optimal solution to the problem is the point in the feasible region with the smallest z value.

An isocost line with the smallest z value passes through point E and is the optimal solution at $x_1 = 3.6$ and $x_2 = 1.4$ giving $z = 320$.

Both the high-income women and high-income men constraints are satisfied, both constraints are binding.

Example 3. Two Mines

min 180x + 160y

st 6x + y ≥ 12

3x + y ≥ 8

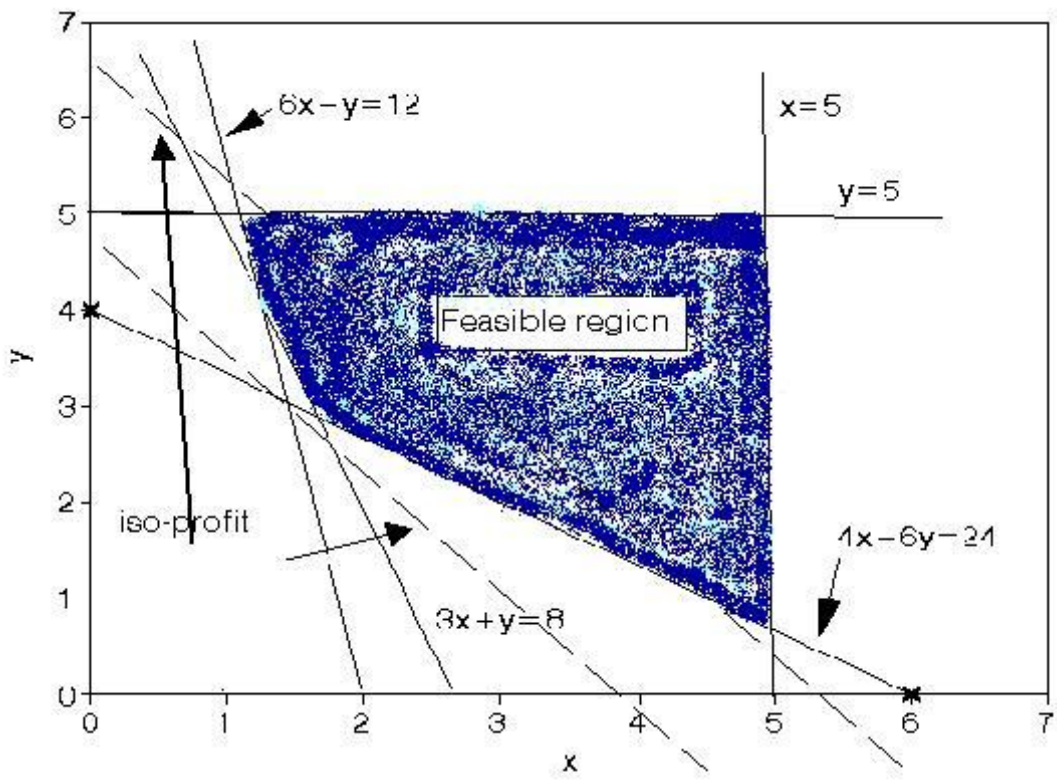
4x + 6y ≥

24 x ≤ 5

y ≤ 5 x,

y ≥ 0

Answer



Optimal sol'n is 765.71. 1.71 days mine X and 2.86 days mine Y are operated.

Example 4. Modified Giapetto

max z = $4x_1 + 2x_2$

s.t.	$2x_1 + x_2$	≤ 100	(Finishing constraint)
	$x_1 + x_2$	≤ 80	(Carpentry constraint)
	x_1	≤ 40	(Demand constraint)
	$x_1, x_2 \geq 0$		(Sign restrictions)

Answer

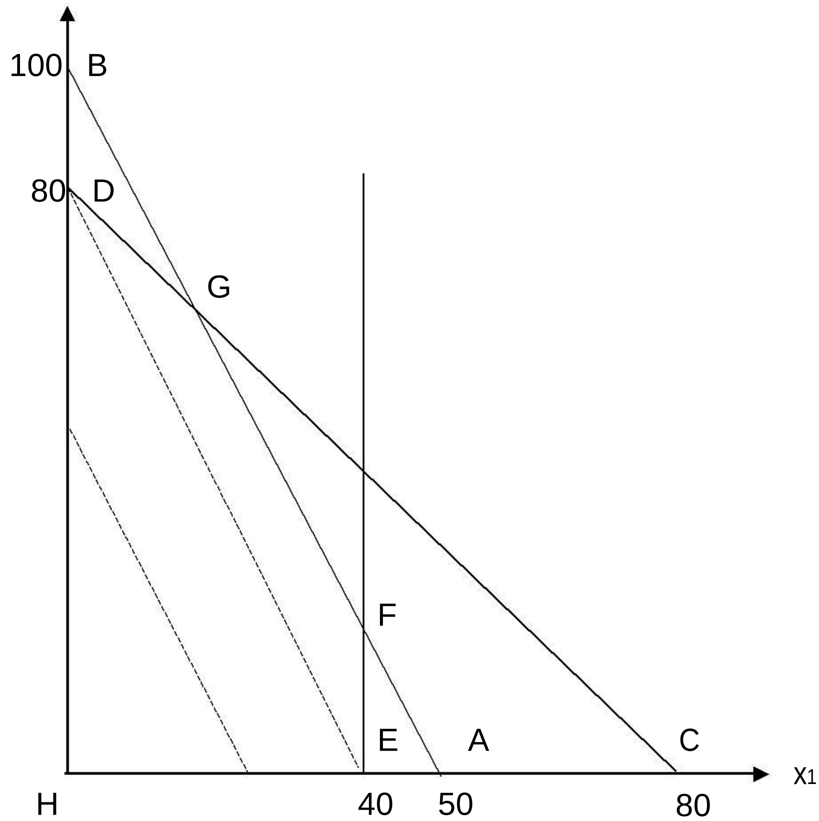
Points on the line between points G (20, 60) and F (40, 20) are the ***alternative optimal solutions*** (see figure below).

Thus, for $0 \leq c \leq 1$,

$$c\,[20\ 60] + (1 - c)\,[40\ 20] = \mathbf{[40 - 20c,\ 20 + 40c]}$$

will be optimal

For all optimal solutions, the optimal objective function value is 200.



Example 5. Modified Giapetto (v. 2)

Add constraint $x_2 \geq 90$ (Constraint on demand for trains).

Answer

No feasible region: ***Infeasible LP***

Example 6. Modified Giapetto (v. 3)

Only use constraint $x_2 \geq 90$

Answer

Isoprofit line never lose contact with the feasible region: ***Unbounded LP***

3.2.3 The Simplex Algorithm

Note that in the examples considered at the graphical solution, the unique optimal solution to the LP occurred at a vertex (corner) of the feasible region. In fact it is true that for *any* LP the optimal solution occurs at a vertex of the feasible region. This fact is the key to the simplex algorithm for solving LP's.

Essentially the simplex algorithm starts at one vertex of the feasible region and moves (at each iteration) to another (adjacent) vertex, improving (or leaving unchanged) the objective function as it does so, until it reaches the vertex corresponding to the optimal LP solution.

The simplex algorithm for solving linear programs (LP's) was developed by Dantzig in the late 1940's and since then a number of different versions of the algorithm have been developed. One of these later versions, called the *revised simplex* algorithm (sometimes known as the "product form of the inverse" simplex algorithm) forms the basis of most modern computer packages for solving LP's.

Steps

1. Convert the LP to standard form
2. Obtain a basic feasible solution (bfs) from the standard form
3. Determine whether the current bfs is optimal. If it is optimal, stop.
4. If the current bfs is not optimal, determine which nonbasic variable should become a basic variable and which basic variable should become a nonbasic variable to find a new bfs with a better objective function value
5. Go back to Step 3.

Related concepts:

Standard form: all constraints are equations and all variables are nonnegative bfs: any basic solution where all variables are nonnegative

Nonbasic variable: a chosen set of variables where variables equal to 0

Basic variable: the remaining variables that satisfy the system of equations at the standard form

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Example 1. Dakota Furniture

(Winston 4.3, p. 134)

Dakota Furniture makes desks, tables, and chairs. Each product needs the limited resources of lumber, carpentry and finishing; as described in the table. At most 5 tables can be sold per week. Maximize weekly revenue.

Resource	Desk	Table	Chair	Max Avail.
Lumber (board ft.)	8	6	1	48
Finishing hours	4	2	1.5	20
Carpentry hours	2	1.5	.5	8
Max Demand	unlimited	5	unlimited	
Price ($)	60	30	20	

LP Model:

Let x_1, x_2, x_3 be the number of desks, tables and chairs produced. Let the weekly profit be \$z. Then, we must

$$
\begin{aligned}
\max\ z &= 60x_1 + 30x_2 + 20x_3 \\
\text{s.t.}\quad & 8x_1 + 6x_2 + x_3 \le 48 \\
& 4x_1 + 2x_2 + 1.5x_3 \le 20 \\
& 2x_1 + 1.5x_2 + .5x_3 \le 8 \\
& x_2 \le 5 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}
$$

Solution with Simplex Algorithm

First introduce slack variables and convert the LP to the standard form and write a canonical form

$$
\begin{array}{llllllllll}
R_0 & z & -60x_1 & -30x_2 & -20x_3 & & & & & = 0 \\
R_1 & & 8x_1 & +6x_2 & +x_3 & +s_1 & & & & = 48 \\
R_2 & & 4x_1 & +2x_2 & +1.5x_3 & & +s_2 & & & = 20 \\
R_3 & & 2x_1 & +1.5x_2 & +.5x_3 & & & +s_3 & & = 8 \\
R_4 & & & x_2 & & & & & +s_4 & = 5
\end{array}
$$

$x_1, x_2, x_3, s_1, s_2, s_3, s_4 \ge 0$

Obtain a starting bfs.

As (x_1, x_2, x_3) = 0 is feasible for the original problem, the below given point where three of the variables equal 0 (the ***non-basic variables***) and the four other variables (the ***basic variables***) are determined by the four equalities is an obvious bfs:

$$x_1 = x_2 = x_3 = 0,\ s_1 = 48,\ s_2 = 20,\ s_3 = 8,\ s_4 = 5.$$

.

Determine whether the current bfs is optimal.

Determine whether there is any way that z can be increased by increasing some nonbasic variable.

If each nonbasic variable has a nonnegative coefficient in the objective function row (***row 0***), current bfs is optimal.

However, here all nonbasic variables have negative coefficients: It is not optimal.

Find a new bfs

z increases most rapidly when x_1 is made non-zero; i.e. x_1 is the ***entering variable***.

Examining R_1, x_1 can be increased only to 6. More than 6 makes $s_1 < 0$. Similarly R_2, R_3, and R_4, give limits of 5, 4, and no limit for x_1 (***ratio test***). The smallest ratio is the largest value of the entering variable that will keep all the current basic variables nonnegative. Thus by R_3, x_1 can only increase to $x_1 = 4$ when s_3 becomes 0. We say s_3 is the ***leaving variable*** and R_3 is the ***pivot equation***.

Now we must rewrite the system so the values of the basic variables can be read off.

The new *pivot equation* (R_3/2) is

$$R_3' : x_1 + .75x_2 + .25x_3 + .5s_3 = 4$$

Then use R_3' to eliminate x_1 in all the other rows.

R0'=R0+60R3', R1'=R1-8R3', R2'=R2-4R3', R4'=R4

R_0'	z		$+ 15x_2$	$- 5x_3$			$+ 30s_3$		$= 240$	$z = 240$
R_1'				$- x_3$	$+ s_1$		$- 4s_3$		$= 16$	$s_1 = 16$
R_2'			$- x_2$	$+ .5x_3$		$+ s_2$	$- 2s_3$		$= 4$	$s_2 = 4$
R_3'		x_1	$+ .75x_2$	$+ .25x_3$			$+ .5s_3$		$= 4$	$x_1 = 4$
R_4'			x_2					$+ s_4$	$= 5$	$s_4 = 5$

The new bfs is $x_2 = x_3 = s_3 = 0$, $x_1 = 4$, $s_1 = 16$, $s_2 = 4$, $s_4 = 5$ making $z = 240$.

Check optimality of current bfs. Repeat steps until an optimal solution is reached

We increase z fastest by making x_3 non-zero (i.e. x_3 enters).
x_3 can be increased to at most $x_3 = 8$, when $s_2 = 0$ (i.e. s_2 leaves.)

Rearranging the pivot equation gives

R_2'' $- 2x_2 + x_3 + 2s_2 - 4s_3 = 8$ ($R_2' \times 2$).

Row operations with R_2'' eliminate x_3 to give the new system

$R_0'' = R_0' + 5R_2''$, $R_1'' = R_1' + R_2''$, $R_3'' = R_3' - .5R_2''$, $R_4'' = R_4'$

The bfs is now $x_2 = s_2 = s_3 = 0$, $x_1 = 2$, $x_3 = 8$, $s_1 = 24$, $s_4 = 5$ making $z = 280$.
Each nonbasic variable has a nonnegative coefficient in row 0 ($5x_2$, $10s_2$, $10s_3$).
THE CURRENT SOLUTION IS OPTIMAL

Report: Dakota furniture's optimum weekly profit would be 280$ if they produce 2 desks and 8 chairs.

This was once written as a tableau.
(***Use tableau format for each operation in all HW and exams!!!***)

$$
\begin{aligned}
\max z &= 60x_1 + 30x_2 + 20x_3 \\
\text{s.t.} \quad & 8x_1 + 6x_2 + x_3 \le 48 \\
& 4x_1 + 2x_2 + 1.5x_3 \le 20 \\
& 2x_1 + 1.5x_2 + .5x_3 \le 8 \\
& x_2 \le 5 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}
$$

Initial tableau:

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV	Ratio
1	-60	-30	-20	0	0	0	0	0	$z = 0$	
0	8	6	1	1	0	0	0	48	$s_1 = 48$	6
0	4	2	1.5	0	1	0	0	20	$s_2 = 20$	5
0	2	1.5	0.5	0	0	1	0	8	$s_3 = 8$	4
0	0	1	0	0	0	0	1	5	$s_4 = 5$	-

First tableau:

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV	Ratio
1	0	15	-5	0	0	30	0	240	$z = 240$	
0	0	0	-1	1	0	-4	0	16	$s_1 = 16$	-
0	0	-1	0.5	0	1	-2	0	4	$s_2 = 4$	8
0	1	0.75	0.25	0	0	0.5	0	4	$x_1 = 4$	16
0	0	1	0	0	0	0	1	5	$s_4 = 5$	-

Second and optimal tableau:

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV	Ratio
1	0	5	0	0	10	10	0	280	$z = 280$	
0	0	-2	0	1	2	-8	0	24	$s_1 = 24$	
0	0	-2	1	0	2	-4	0	8	$x_3 = 8$	
0	1	1.25	0	0	-0.5	1.5	0	2	$x_1 = 2$	
0	0	1	0	0	0	0	1	5	$s_4 = 5$	

Example 2. Modified Dakota Furniture

Dakota example is modified: $35/table

new $z = 60\,x_1 + 35\,x_2 + 20\,x_3$

Second and optimal tableau for the modified problem:

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV	Ratio
1	0	**0**	0	0	10	10	0	280	z=280	
0	0	-2	0	1	2	-8	0	24	s_1=24	-
0	0	-2	1	0	2	-4	0	8	x_3=8	-
0	1	1.25	0	0	-0.5	1.5	0	2	x_1=2	2/1.25
0	0	1	0	0	0	0	1	5	s_4=5	5/1

Another optimal tableau for the modified problem:

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV
1	0	0	0	0	10	10	0	280	z=280
0	1.6	0	0	1	1.2	-5.6	0	27.2	s_1=27.2
0	1.6	0	1	0	1.2	-1.6	0	11.2	x_3=11.2
0	0.8	1	0	0	-0.4	1.2	0	1.6	x_2=1.6
0	-0.8	0	0	0	0.4	-1.2	1	3.4	s_4=3.4

Therefore the optimal solution is as follows:

z = 280 and for $0 \le c \le 1$

$$\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = c \begin{vmatrix} 2 \\ 0 \\ 8 \end{vmatrix} + (1 - c) \begin{vmatrix} 0 \\ 1.6 \\ 11.2 \end{vmatrix} = \begin{vmatrix} 2c \\ 1.6 - 1.6c \\ 11.2 - 3.2c \end{vmatrix}$$

Example 3. Unbounded LPs

z	x_1	x_2	x_3	s_1	s_2	z	RHS	BV	Ratio
1	0	2	**-9**	0	12	4	100	z=100	
0	0	1	-6	1	6	-1	20	x_4=20	None
0	1	1	-1	0	1	0	5	x_1=5	None

Since ratio test fails, the LP under consideration is an unbounded LP.

3.2.4 The Big M Method

If an LP has any ≥ or = constraints, a starting bfs may not be readily apparent.

When a bfs is not readily apparent, the Big M method or the two-phase simplex method may be used to solve the problem.

The Big M method is a version of the Simplex Algorithm that first finds a bfs by adding "artificial" variables to the problem. The objective function of the original LP must, of course, be modified to ensure that the artificial variables are all equal to 0 at the conclusion of the simplex algorithm.

Steps

1. Modify the constraints so that the RHS of each constraint is nonnegative (This requires that each constraint with a negative RHS be multiplied by -1. Remember that if you multiply an inequality by any negative number, the direction of the inequality is reversed!). After modification, identify each constraint as a ≤, ≥ or = constraint.
2. Convert each inequality constraint to standard form (If constraint i is a ≤ constraint, we add a slack variable s_i; and if constraint i is a ≥ constraint, we subtract an excess variable e_i).
3. Add an artificial variable a_i to the constraints identified as ≥ or = constraints at the end of Step 1. Also add the sign restriction $a_i \geq 0$.
4. Let M denote a very large positive number. If the LP is a min problem, add (for each artificial variable) Ma_i to the objective function. If the LP is a max problem, add (for each artificial variable) $-Ma_i$ to the objective function.
5. Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from row 0 before beginning the simplex. Now solve the transformed problem by the simplex (In choosing the entering variable, remember that M is a very large positive number!).

If all artificial variables are equal to zero in the optimal solution, we have found the **optimal solution** to the original problem.

If any artificial variables are positive in the optimal solution, the original problem is **infeasible**!!!

Example 1. Oranj Juice

Bevco manufactures an orange flavored soft drink called Oranj by combining orange soda and orange juice. Each ounce of orange soda contains 0.5 oz of sugar and 1 mg of vitamin C. Each ounce of orange juice contains 0.25 oz of sugar and 3 mg of vitamin C. It costs Bevco 2¢ to produce an ounce of orange soda and 3¢ to produce an ounce of orange juice. Marketing department has decided that each 10 oz bottle of Oranj must contain at least 20 mg of vitamin C and at most 4 oz of sugar. Use LP to determine how Bevco can meet marketing dept.'s requirements at minimum cost.

LP Model:

Let x_1 and x_2 be the quantity of ounces of orange soda and orange juice (respectively) in a bottle of Oranj.

$$
\begin{array}{lll}
\min z = 2x_1 + 3x_2 & & \\
\text{s.t.} & 0.5\,x_1 + 0.25\,x_2 \le 4 & \text{(sugar const.)} \\
 & x_1 + 3\,x_2 \ge 20 & \text{(vit. C const.)} \\
 & x_1 + x_2 = 10 & \text{(10 oz in bottle)} \\
 & x_1, x_2 \ge 0 &
\end{array}
$$

Solving Oranj Example with Big M Method

1. Modify the constraints so that the RHS of each constraint is nonnegative The RHS of each constraint is nonnegative

2. Convert each inequality constraint to standard form

$$
\begin{array}{l}
z - 2x_1 - 3x_2 = 0 \\
0.5x_1 + 0.25x_2 + s_1 = 4 \\
x_1 + 3x_2 - e_2 = 20 \\
x_1 + x_2 = 10
\end{array}
$$

all variables nonnegative

3. Add a_i to the constraints identified as > or = const.s

$$
\begin{array}{ll}
z - 2x_1 - 3x_2 = 0 & \text{Row 0} \\
0.5x_1 + 0.25x_2 + s_1 = 4 & \text{Row 1} \\
x_1 + 3x_2 - e_2 + a_2 = 20 & \text{Row 2} \\
x_1 + x_2 + a_3 = 10 & \text{Row 3}
\end{array}
$$

all variables nonnegative

4. Add Ma_i to the objective function (min problem)

min $z = 2x_1 + 3x_2 + Ma_2 +$
Ma_3 Row 0 will change to

$z - 2x_1 - 3x_2 - Ma_2 - Ma_3 = 0$

5. Since each artificial variable are in our starting bfs, they must be eliminated from row 0

New Row 0 = Row 0 + M * Row 2 + M * Row 3

$z + (2M-2)\,x_1 + (4M-3)\,x_2 - M\,e_2 = 30M$ New Row 0

Initial tableau:

z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	BV	Ratio
1	2M-2	**4M-3**	0	-M	0	0	30M	z=30M	
0	0.5	0.25	1	0	0	0	4	s_1=4	16
0	1	$\boxed{3}$	0	-1	1	0	20	a_2=20	**20/3***
0	1	1	0	0	0	1	10	a_3=10	10

In a min problem, entering variable is the variable that has the "most positive" coefficient in row 0!

First tableau:

z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	BV	Ratio
1	**(2M-3)/3**	0	0	(M-3)/3	(3-4M)/3	0	20+3.3Mz		
0	5/12	0	1	1/12	-1/12	0	7/3	s_1	28/5
0	1/3	1	0	-1/3	1/3	0	20/3	x_2	20
0	$\boxed{2/3}$	0	0	1/3	-1/3	1	10/3	a_3	**5***

Optimal tableau:

z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	BV
1	0	0	0	-1/2	(1-2M)/2	(3-2M)/2	25	z=25
0	0	0	1	-1/8	1/8	-5/8	1/4	s_1=1/4
0	0	1	0	-1/2	1/2	-1/2	5	x_2=5
0	1	0	0	1/2	-1/2	3/2	5	x_1=5

Report:

In a bottle of Oranj, there should be 5 oz orange soda and 5 oz orange juice.

In this case the cost would be 25¢.

Example 2. Modified Oranj Juice

Consider Bevco's problem. It is modified so that 36 mg of vitamin C are required.

Related LP model is given as follows:

Let x_1 and x_2 be the quantity of ounces of orange soda and orange juice (respectively) in a bottle of Oranj.

$$
\begin{array}{llll}
\min & z = 2x_1 + 3x_2 & & \\
\text{s.t.} & 0.5\, x_1 + 0.25\, x_2 \le 4 & & \text{(sugar const.)} \\
 & x_1 + 3\, x_2 \ge 36 & & \text{(vit. C const.)} \\
 & x_1 + x_2 = 10 & & \text{(10 oz in bottle)} \\
 & x_1, x_2 \ge 0 & &
\end{array}
$$

Solving with Big M method:

Initial tableau:

z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	BV	Ratio
1	2M-2	**4M-3**	0	-M	0	0	46M	z=46M	
0	0.5	0.25	1	0	0	0	4	s_1=4	16
0	1	3	0	-1	1	0	36	a_2=36	36/3
0	1	$\boxed{\mathbf{1}}$	0	0	0	1	10	a_3=10	**10**

Optimal tableau:

z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	BV
1	1-2M	0	0	-M	0	3-4M	30+6M	z=30+6M
0	1/4	0	1	0	0	-1/4	3/2	s_1=3/2
0	-2	0	0	-1	1	-3	6	a_2=6
0	1	1	0	0	0	1	10	x_2=10

An artificial variable (a_2) is BV so the original LP has no feasible solution

Report:

It is impossible to produce Oranj under these conditions.

3.3 DUALITY

3.3.1 Primal – Dual

Associated with any LP is another LP called the ***dual***. Knowledge of the dual provides interesting economic and sensitivity analysis insights. When taking the dual of any LP, the given LP is referred to as the ***primal***. If the primal is a max problem, the dual will be a min problem and vice versa.

3.3.2 Finding the Dual of an LP

The dual of a ***normal max*** problem is a ***normal min*** problem.

Normal max problem is a problem in which all the variables are required to be nonnegative and all the constraints are ≤ constraints.

Normal min problem is a problem in which all the variables are required to be nonnegative and all the constraints are ≥ constraints.

Similarly, the dual of a normal min problem is a normal max problem.

Finding the Dual of a Normal Max Problem

PRIMAL

max z = $c_1x_1 + c_2x_2 + \ldots + c_nx_n$

s.t. $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2$

$\ldots \quad \ldots \quad \ldots \quad \ldots$

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m$

$x_j \ge 0$ ($j = 1, 2, \ldots, n$)

DUAL

min w = $b_1y_1 + b_2y_2 + \ldots + b_my_m$

s.t. $a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m \ge c_1$

$a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \ge c_2$

$\ldots \quad \ldots \quad \ldots \quad \ldots$

$a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \ge c_n$

$y_i \ge 0$ ($i = 1, 2, \ldots, m$)

Finding the Dual of a Normal Min Problem

PRIMAL

min w = $b_1y_1 + b_2y_2 + \ldots + b_my_m$

s.t. $a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m \ge c_1$

$a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \ge c_2$

$\ldots \quad \ldots \quad \ldots \quad \ldots$

$a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \ge c_n$

$y_i \ge 0$ ($i = 1, 2, \ldots, m$)

DUAL

max z = $c_1x_1 + c_2x_2 + \ldots + c_nx_n$

s.t. $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \le b_1$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \le b_2$

$\dots \quad \dots \quad \dots \quad \dots$

$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \le b_m$

$x_j \ge 0$ ($j = 1, 2, \dots, n$)

Finding the Dual of a Nonnormal Max Problem

If the *i*th primal constraint is a ≥ constraint, the corresponding dual variable y_i must satisfy $y_i \le 0$

If the *i*th primal constraint is an equality constraint, the dual variable y_i is now unrestricted in sign (urs).

If the *i*th primal variable is urs, the *i*th dual constraint will be an equality constraint

Finding the Dual of a Nonnormal Min Problem

If the *i*th primal constraint is a ≤ constraint, the corresponding dual variable x_i must satisfy $x_i \le 0$

If the *i*th primal constraint is an equality constraint, the dual variable x_i is now urs.

If the *i*th primal variable is urs, the *i*th dual constraint will be an equality constraint

3.3.3 The Dual Theorem

The primal and dual have equal optimal objective function values (if the problems have optimal solutions).

Weak duality implies that if for any feasible solution to the primal and an feasible solution to the dual, the *w*-value for the feasible dual solution will be at least as large as the *z*-value for the feasible primal solution → $z \le w$.

Consequences

Any feasible solution to the dual can be used to develop a bound on the optimal value of the primal objective function.

If the primal is unbounded, then the dual problem is infeasible. If the dual is unbounded, then the primal is infeasible.

How to read the optimal dual solution from Row 0 of the optimal tableau if the primal is a max problem:

'optimal value of dual variable y_i'

= 'coefficient of s_i in optimal row 0' (if const. i is a ≤ const.)

= –'coefficient of e_i in optimal row 0' (if const. i is a ≥ const.)

= 'coefficient of a_i in optimal row 0' – M (if const. i is a = const.)

How to read the optimal dual solution from Row 0 of the optimal tableau if the primal is a min problem:

'optimal value of dual variable x_i'

= 'coefficient of s_i in optimal row 0' (if const. i is a ≤ const.)

= –' coefficient of e_i in optimal row 0' (if const. i is a ≥ const.)

= 'coefficient of a_i in optimal row 0' + M (if const. i is a = const.)

3.3.4 Economic Interpretation

When the primal is a normal max problem, the dual variables are related to the value of resources available to the decision maker. For this reason, dual variables are often referred to as ***resource shadow prices***.

Example

PRIMAL

Let x_1, x_2, x_3 be the number of desks, tables and chairs produced. Let the weekly profit be \$$z$. Then, we must

$$
\begin{aligned}
\max\ z &= 60x_1 + 30x_2 + 20x_3 \\
\text{s.t.}\quad & 8x_1 + 6x_2 + x_3 \le 48 \text{ (Lumber constraint)} \\
& 4x_1 + 2x_2 + 1.5x_3 \le 20 \text{ (Finishing hour constraint)} \\
& 2x_1 + 1.5x_2 + 0.5x_3 \le 8 \text{ (Carpentry hour constraint)} \\
& x_1, x_2, x_3 \ge 0
\end{aligned}
$$

DUAL

Suppose an entrepreneur wants to purchase all of Dakota's resources.

In the dual problem y_1, y_2, y_3 are the resource prices (price paid for one board ft of lumber, one finishing hour, and one carpentry hour).

\$w is the cost of purchasing the resources.

Resource prices must be set high enough to induce Dakota to sell. i.e. total purchasing cost equals total profit.

$$
\begin{aligned}
\min w = {} & 48y_1 + 20y_2 + 8y_3 \\
\text{s.t.} \quad & 8y_1 + 4y_2 + 2y_3 \geq 60 \text{ (Desk constraint)} \\
& 6y_1 + 2y_2 + 1.5y_3 \geq 30 \text{ (Table constraint)} \\
& y_1 + 1.5y_2 + 0.5y_3 \geq 20 \text{ (Chair constraint)} \\
& y_1, y_2, y_3 \geq 0
\end{aligned}
$$

3.4 SENSITIVITY ANALYSIS

3.4.1 Reduced Cost

For any nonbasic variable, the reduced cost for the variable is the amount by which the nonbasic variable's objective function coefficient must be improved before that variable will become a basic variable in some optimal solution to the LP.

If the objective function coefficient of a nonbasic variable x_k is improved by its reduced cost, then the LP will have alternative optimal solutions at least one in which x_k is a basic variable, and at least one in which x_k is not a basic variable.

If the objective function coefficient of a nonbasic variable x_k is improved by more than its reduced cost, then any optimal solution to the LP will have x_k as a basic variable and $x_k > 0$.

Reduced cost of a basic variable is zero (see definition)!

3.4.2 Shadow Price

We define the shadow price for the *i*th constraint of an LP to be the amount by which the optimal z value is "improved" (increased in a max problem and decreased in a min problem) if the RHS of the *i*th constraint is increased by 1.

This definition applies only if the change in the RHS of the constraint leaves the current basis optimal!

A ≥ constraint will always have a nonpositive shadow price; a ≤ constraint will always have a nonnegative shadow price.

3.4.3 Conceptualization

$$\max z = 5x_1 + x_2 + 10x_3$$

$$x_1 + x_3 \leq$$

$$100 x_2 \leq 1$$

All variables ≥ 0

This is a very easy LP model and can be solved manually without utilizing Simplex. $x_2 = 1$ (This variable does not exist in the first constraint. In this case, as the problem is a maximization problem, the optimum value of the variable equals the RHS value of the second constraint).

$x_1 = 0$, $x_3 = 100$ (These two variables do exist only in the first constraint and as the objective function coefficient of x_3 is greater than that of x_1, the optimum value of x_3 equals the RHS value of the first constraint).

Hence, the optimal solution is as follows: $z = 1001$, $[x_1, x_2, x_3] = [0, 1, 100]$

Similarly, sensitivity analysis can be executed manually.

Reduced Cost

As x_2 and x_3 are in the basis, their reduced costs are 0.

In order to have x_1 enter in the basis, we should make its objective function coefficient as great as that of x_3. In other words, improve the coefficient as 5 (10-5). New objective function would be (max $z = 10x_1 + x_2 + 10x_3$) and there would be at least two optimal solutions for $[x_1, x_2, x_3]$: [0, 1, 100] and [100, 1, 0].

Therefore reduced cost of x_1 equals 5.

If we improve the objective function coefficient of x_1 more than its reduced cost, there would be a unique optimal solution: [100, 1, 0].

Shadow Price

If the RHS of the first constraint is increased by 1, new optimal solution of x_3 would be 101 instead of 100. In this case, new z value would be 1011.

If we use the definition: 1011 - 1001 = 10 is the shadow price of the first constraint.

Similarly the shadow price of the second constraint can be calculated as 1 (please find it).

3.4.5 Some important equations

If the change in the RHS of the constraint leaves the current basis optimal (within the allowable RHS range), the following equations can be used to calculate new objective function value:

for maximization problems

new obj. fn. value = old obj. fn. value + (new RHS – old RHS) × shadow price for minimization problems

new obj. fn. value = old obj. fn. value – (new RHS – old RHS) × shadow price

For Lindo example, as the allowable increases in RHS ranges are infinity for each constraint, we can increase RHS of them as much as we want. But according to allowable decreases, RHS of the first constraint can be decreased by 100 and that of second constraint by 1.

Lets assume that new RHS value of the first constraint is 60.

As the change is within allowable range, we can use the first equation (max. problem):

$$z_{new} = 1001 + (60 - 100)\ 10 = 601.$$

3.4.6 Utilizing Simplex for Sensitivity

In Dakota furniture example; x_1, x_2, and x_3 were representing the number of desks, tables, and chairs produced.

The LP formulated for profit maximization:

$$\begin{array}{llllllllll}
\max z = & 60\,x_1 & 30\,x_2 & 20x_3 & & & & & & \\
& 8\,x_1 & +\ 6\,x_2 & +\ x_3 & +\ s_1 & & & & = 48 & \text{Lumber} \\
& 4\,x_1 & +\ 2\,x_2 & +1.5\,x_3 & & +\ s_2 & & & = 20 & \text{Finishing} \\
& 2\,x_1 & +1.5\,x_2 & +.5\,x_3 & & & +\ s_3 & & = 8 & \text{Carpentry} \\
& & x_2 & & & & & +\ s_4 & = 5 & \text{Demand}
\end{array}$$

The optimal solution was:

$$\begin{array}{lllllllll}
z & & +5\,x_2 & & & +10\,s_2 & +10\,s_3 & & = 280 \\
& & -2\,x_2 & & +s_1 & +2\,s_2 & -8\,s_3 & & = 24 \\
& & -2\,x_2 & +\ x_3 & & +2\,s_2 & -4\,s_3 & & = 8 \\
& +\ x_1 & +\ 1.25\,x_2 & & & -.5\,s_2 & +1.5\,s_3 & & = 2 \\
& & x_2 & & & & & +\ s_4 & = 5
\end{array}$$

Analysis 1

Suppose available finishing time changes from 20   20+ , then we have the system:

$$\begin{array}{lllllllll}
z' = & 60\,x_1' & +\ 30\,x_2' & +\ 20\,x_3' & & & & & \\
& 8\,x_1' & +\ 6\,x_2' & +\ x_3' & +\ s_1' & & & & = 48 \\
& 4\,x_1' & +\ 2\,x_2' & +1.5\,x_3' & & +\ s_2' & & & = 20+ \\
& 2\,x_1' & +1.5\,x_2' & +\ .5\,x_3' & & & +\ s_3' & & = 8 \\
& & +\ x_2' & & & & & +\ s_4' & = 5
\end{array}$$

or equivalently:

$$
\begin{array}{rrrrrrrl}
z' = & 60\,x_1' & +\,30\,x_2' & +\,20\,x_3' & & & & \\
 & 8\,x_1' & +\ 6\,x_2' & +\ \ x_3' & +\,s_1' & & & = 48 \\
 & 4\,x_1' & +\ 2\,x_2' & +1.5\,x_3' & & +(s_2'-\ ) & & = 20 \\
 & 2\,x_1' & +1.5\,x_2' & +\ .5\,x_3' & & & +\,s_3' & = 8 \\
 & & +\ \ x_2' & & & & +\,s_4' & = 5
\end{array}
$$

That is z , x_1 , x_2 , x_3 ,x_4 ,s_1 ,s_2 - ,s_3 ,s_4 satisfy the original problem, and hence

(1) Substituting in:

$$
\begin{array}{lrrrrrrl}
z' & & +5\,x_2' & & & +10(s_2'-\ ) & +10\,s_3' & = 280 \\
 & & -2\,x_2' & & +\,s_1' & +2(s_2'-\ ) & -8\,s_3' & = \ 24 \\
 & & -2\,x_2' & +\,x_3' & & +2(s_2'-\ ) & -4\,s_3' & = \ 8 \\
 & +\,x_1' & +1.25\,x_2' & & & -.5(s_2'-\ ) & +1.5\,s_3' & = \ 2 \\
 & & x_2' & & & & +\,s_4' & = \ 5
\end{array}
$$

and thus

$$
\begin{array}{lrrrrrrl}
z' & & +5\,x_2' & & & +10\,s_2' & +10\,s_3' & = 280+10 \\
 & & -2\,x_2' & & +s_1' & +2\,s_2' & -8\,s_3' & = \ 24+2 \\
 & & -2\,x_2' & +\,x_3' & & +2\,s_2' & -4\,s_3' & = \ 8+2 \\
 & +\,x_1' & +1.25\,x_2' & & & -.5\,s_2' & +1.5\,s_3' & = \ 2-.5 \\
 & & x_2' & & & & +\,s_4' & = \ 5
\end{array}
$$

For -4 4, the new system maximizes z'. In this range RHS values are non-negative.

As increases, revenue increases by 10 . Therefore, the ***shadow price*** of finishing labor is \$10 per hr. (This is valid for up to 4 extra hours or 4 fewer hours).

Analysis 2

What happens if revenue from desks changes to \$60+ ? For small revenue increases by 2 (as we are making 2 desks currently). But how large an increase is possible?

The new revenue is:

$$z' = (60+\ )x_1+30x_2+20x_3 = z+\ x_1$$

$$= (280 - 5x_2 - 10s_2 - 10s_3) + \ (2 - 1.25x_2 + .5s_2 - 1.5s_3)$$

$$= 280 + 2 - (5 + 1.25\ )x_2 - (10-.5\ )s_2 - (10 + 1.5\ )s_3$$

So the top line in the final system would be:

$$z' + (5 + 1.25\ )x_2 + (10 - .5\ )s_2 + (10 + 1.5\ )s_3 = 280 + 2$$

Provided all terms in this row are we are still optimal.

For -4 20, the current production schedule is still optimal.

Analysis 3

If revenue from a non-basic variable changes, the revenue is

$$z' = 60x_1 + (30 + \ )x_2 + 20x_3 = z + x_2$$

$$= 280 - 5x_2 - 10s_2 - 10s_3 + \ x_2$$

$$= 280 - (5 - \ )x_2 - 10s_2 - 10s_3$$

The current solution is optimal for 5. But when 5 or the revenue per table is increased past \$35, it becomes better to produce tables. We say the **reduced cost** of tables is \$5.00.

4. TRANSPORTATION PROBLEMS

4.1 FORMULATING TRANSPORTATION PROBLEMS

In general, a transportation problem is specified by the following information:

A set of m ***supply points*** from which a good/service is shipped. Supply point i can supply at most s_i units.

A set of n ***demand points*** to which the good/service is shipped. Demand point j must receive at least d_j units.

Each unit produced at supply point i and shipped to demand point j incurs a variable cost of c_{ij}.

The relevant data can be formulated in a ***transportation tableau***:

	Demand point 1	Demand point 2	.....	Demand point n	SUPPLY
Supply point 1	c_{11}	c_{12}		c_{1n}	s_1
Supply point 2	c_{21}	c_{22}		c_{2n}	s_2
.....					
Supply point m	c_{m1}	c_{m2}		c_{mn}	s_m
DEMAND	d_1	d_2		d_n	

If total supply equals total demand then the problem is said to be a ***balanced transportation problem***.

Let x_{ij} = number of units shipped from supply point i to demand point j

- Decision variable x_{ij}: number of units shipped from supply point i to demand point j

then the general LP representation of a transportation problem is

$$\min \sum_{i}\sum_{j} c_{ij}x_{ij}$$

s.t. $\sum_{j} x_{ij} \le s_i \; (i=1,2,\ldots,m)$ Supply constraints

$\sum_{i} x_{ij} \ge d_j \; (j=1,2,\ldots,n)$ Demand constraints

$x_{ij} \geq 0$

If a problem has the constraints given above and is a *maximization* problem, it is still a transportation problem.

4.1.1 Formulating Balanced Transportation Problem

Example 1. Powerco

Powerco has three electric power plants that supply the needs of four cities. Each power plant can supply the following numbers of kwh of electricity: plant 1, 35 million; plant 2, 50 million; and plant 3, 40 million. The peak power demands in these cities as follows (in kwh): city 1, 45 million; city 2, 20 million; city 3, 30 million; city 4, 30 million. The costs of sending 1 million kwh of electricity from plant to city is given in the table below. To minimize the cost of meeting each city's peak power demand, formulate a balanced transportation problem in a transportation tableau and represent the problem as a LP model.

	To			
From	City 1	City 2	City 3	City 4
Plant 1	\$8	\$6	\$10	\$9
Plant 2	\$9	\$12	\$13	\$7
Plant 3	\$14	\$9	\$16	\$5

Answer

<u>Representation of the problem as a LP model</u>

x_{ij}: number of (million) kwh produced at plant *i* and sent to city *j*.

min z = 8 x11 + 6 x12 + 10 x13 + 9 x14 + 9 x21 + 12 x22 + 13 x23 + 7 x24 + 14 x31 + 9 x32 + 16 x33 + 5 x34

s.t. x11 + x12 + x13 + x14 $\leq$ 35 (supply constraints)

x21 + x22 + x23 + x24 $\leq$ 50

x31 + x32 + x33 + x34 $\leq$ 40

x11 + x21 + x31 $\geq$ 45 (demand constraints)

x12 + x22 + x32 $\geq$ 20

x13 + x23 + x33 $\geq$ 30

x14 + x24 + x34 $\geq$ 30

$x_{ij} \geq 0$ (i = 1, 2, 3; j = 1, 2, 3, 4)

Formulation of the transportation problem

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8	6	10	9	35
Plant 2	9	12	13	7	50
Plant 3	14	9	16	5	40
DEMAND	45	20	30	30	125

Total supply & total demand both equal 125: "balanced transport'n problem".

4.1.2 Balancing an Unbalanced Transportation Problem

Excess Supply

If total supply exceeds total demand, we can balance a transportation problem by creating a ***dummy demand point*** that has a demand equal to the amount of excess supply. Since shipments to the dummy demand point are not real shipments, they are assigned a cost of zero. These shipments indicate unused supply capacity.

Unmet Demand

If total supply is less than total demand, actually the problem has no feasible solution. To solve the problem it is sometimes desirable to allow the possibility of leaving some demand unmet. In such a situation, *a penalty is often associated with unmet demand.* This means that a ***dummy supply point*** should be introduced.

Example 2. Modified Powerco for Excess Supply

Suppose that demand for city 1 is 40 million kwh. Formulate a balanced transportation problem.

Answer

Total demand is 120, total supply is 125.

To balance the problem, we would add a dummy demand point with a demand of 125 – 120 = 5 million kwh.

From each plant, the cost of shipping 1 million kwh to the dummy is
0. For details see Table 4.

Table 4. Transportation Tableau for Excess Supply

	City 1	City 2	City 3	City 4	Dummy	SUPPLY
Plant 1	8	6	10	9	0	35
Plant 2	9	12	13	7	0	50
Plant 3	14	9	16	5	0	40
DEMAND	40	20	30	30	5	125

Example 3. Modified Powerco for Unmet Demand

Suppose that demand for city 1 is 50 million kwh. For each million kwh of unmet demand, there is a penalty of 80$. Formulate a balanced transportation problem.

Answer

We would add a dummy supply point having a supply of 5 million kwh representing shortage.

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8	6	10	9	35
Plant 2	9	12	13	7	50
Plant 3	14	9	16	5	40
Dummy (Shortage)	80	80	80	80	5
DEMAND	50	20	30	30	130

4.2 FINDING BFS FOR TRANSPORT'N PROBLEMS

For a balanced transportation problem, general LP representation may be written as:

min $_i \;_j c_{ij} x_{ij}$

s.t. $_j x_{ij} = s_i \; (i=1,2, ..., m)$ Supply constraints

$_i x_{ij} = d_j \; (j=1,2, ..., n)$ Demand constraints

$x_{ij} \geq 0$

To find a bfs to a balanced transportation problem, we need to make the following important observation:

If a set of values for the xij's satisfies all but one of the constraints of a balanced transportation problem, the values for the xij's will automatically satisfy the other constraint.

This observation shows that when we solve a balanced transportation, we may omit from consideration any one of the problem's constraints and solve an LP having $m+n-1$ constraints. We arbitrarily assume that the first supply constraint is omitted from consideration. In trying to find a bfs to the remaining $m+n-1$ constraints, you might think that any collection of $m+n-1$ variables would yield a basic solution. But this is not the case: If the $m+n-1$ variables yield a basic solution, the cells corresponding to this set contain ***no loop***.

An ordered sequence of at least four different cells is called a loop if

Any two consecutives cells lie in either the same row or same column No three consecutive cells lie in the same row or column

The last cell in the sequence has a row or column in common with the first cell in the sequence

There are three methods that can be used to find a bfs for a balanced transportation problem:

1. Northwest Corner method
2. Minimum cost method
3. Vogel's method

4.2.1 Northwest Corner Method

We begin in the upper left corner of the transportation tableau and set x_{11} as large as possible (clearly, x_{11} can be no larger than the smaller of s_1 and d_1).

If $x_{11}=s_1$, cross out the first row of the tableau. Also change d_1 to d_1-s_1.

If $x_{11}=d_1$, cross out the first column of the tableau. Change s_1 to s_1-d_1.

If $x_{11}=s_1=d_1$, cross out either row 1 or column 1 (but not both!).

- If you cross out row, change d_1 to 0.
- If you cross out column, change s_1 to 0.

Continue applying this procedure to the most northwest cell in the tableau that does not lie in a crossed out row or column.

Eventually, you will come to a point where there is only one cell that can be assigned a value. Assign this cell a value equal to its row or column demand, and cross out both the cell's row or column.

A bfs has now been obtained.

Example 1.

For example consider a balanced transportation problem given below (We omit the costs because they are not needed to find a bfs!).

				5
				1
				3
2	4	2	1	

Total demand equals total supply (9): this is a balanced transport'n problem.

2				3
				1
				3
X	4	2	1	

2	3			X
				1
				3
X	1	2	1	

2	3			X
	1			X
				3
X	0	2	1	

2	3			X
	1			X
	0	2	1	3
X	0	2	1	

NWC method assigned values to m+n-1 (3+4-1 = 6) variables. The variables chosen by NWC method can not form a loop, so a bfs is obtained.

4.2.2 Minimum Cost Method

Northwest Corner method does not utilize shipping costs, so it can yield an initial bfs that has a very high shipping cost. Then determining an optimal solution may require several pivots.

To begin the minimum cost method, find the variable with the smallest shipping cost (call it x_{ij}). Then assign x_{ij} its largest possible value, min {s_i, d_j}.

As in the NWC method, cross out row i or column j and reduce the supply or demand of the noncrossed-out of row or column by the value of x_{ij}.

Continue like NWC method (instead of assigning upper left corner, the cell with the minimum cost is assigned). See Northwest Corner Method for the details!

Example 2.

2	3	5	6	5
2	1	3	5	10
3	8	4	6	15
12	8	4	6	

2	3	5	6	5
2	8 **1**	3	5	2
3	8	4	6	15
12	X	4	6	

2	3	5	6	5
2 2	1 8	3	5	X
3	8	4	6	15
10	X	4	6	

2 5	3	5	6	X
2 2	1 8	3	5	X
3	8	4	6	15
5	X	4	6	

2 5	3	5	6	X
2 2	1 8	3	5	X
3 5	8	4 4	6 6	10
5	X	4	6	

4.2.3 Vogel's Method

Begin by computing for each row and column a penalty equal to the difference between the two smallest costs in the row and column. Next find the row or column with the largest penalty. Choose as the first basic variable the variable in this row or column that has the smallest cost. As described in the NWC method, make this variable as large as possible, cross out row or column, and change the supply or demand associated with the basic variable (See Northwest Corner Method for the details!). Now recomputed new penalties (using only cells that do not lie in a crossed out row or column), and repeat the procedure until only one uncrossed cell remains. Set this variable equal to the supply or demand associated with the variable, and cross out the variable's row and column.

Example 3.

				Supply	Row penalty
	$\boxed{6}$	$\boxed{7}$	$\boxed{8}$	10	7-6=1
	$\boxed{15}$	$\boxed{80}$	$\boxed{78}$	15	78-15=63
Demand	15	5	5		
Column penalty	15-6=9	80-7=**73**	78-8=70		

				Supply	Row penalty
	$\boxed{6}$	$\boxed{\mathbf{7}}$ 5	$\boxed{8}$	5	8-6=2
	$\boxed{15}$	$\boxed{80}$	$\boxed{78}$	15	78-15=63
Demand	15	X	5		
Column penalty	15-6=9	-	78-8=**70**		

				Supply	Row penalty
	$\boxed{6}$	$\boxed{7}$ 5	$\boxed{\mathbf{8}}$ 5	X	-
	$\boxed{15}$	$\boxed{80}$	$\boxed{78}$	15	-
Demand	15	X	0		
Column penalty	15-6=9	-	-		

	$\boxed{6}$	$\boxed{7}$ 5	$\boxed{8}$ 5	X
	$\boxed{15}$ 15	$\boxed{80}$	$\boxed{78}$ 0	15
Demand	15	X	0	

4.3 THE TRANSPORTATION SIMPLEX METHOD

Steps of the Method

1. If the problem is unbalanced, balance it
2. Use one of the methods to find a bfs for the problem
3. Use the fact that $u_1 = 0$ and $u_i + v_j = c_{ij}$ for all basic variables to find the u's and v's for the current bfs.
4. If $u_i + v_j - c_{ij} \leq 0$ for all nonbasic variables, then the current bfs is optimal. If this is not the case, we enter the variable with the most positive $u_i + v_j - c_{ij}$ into the basis using the *pivoting procedure*. This yields a new bfs. Return to Step 3.

For a maximization problem, proceed as stated, but replace Step 4 by the following step:

If $u_i + v_j - c_{ij} \geq 0$ for all nonbasic variables, then the current bfs is optimal. Otherwise, enter the variable with the most negative $u_i + v_j - c_{ij}$ into the basis using the *pivoting procedure*. This yields a new bfs. Return to Step 3.

Pivoting procedure

1. Find the loop (there is only one possible loop!) involving the entering variable (determined at step 4 of the transport'n simplex method) and some or all of the basic variables.
2. Counting *only cells in the loop*, label those that are an even number (0, 2, 4, and so on) of cells away from the entering variable as *even cells*. Also label those that are an odd number of cells away from the entering variable as *odd cells*.
3. Find the odd cell whose variable assumes the smallest value. Call this value . The variable corresponding to this odd cell will leave the basis. To perform the pivot, decrease the value of each odd cell by and increase the value of each even cell by . The values of variables not in the loop remain unchanged. The pivot is now complete. If = 0, the entering variable will equal 0, and odd variable that has a current value of 0 will leave the basis.

Example 1. Powerco

The problem is balanced (total supply equals total demand).

When the NWC method is applied to the Powerco example, the bfs in the following table is obtained (check: there exist $m+n-1=6$ basic variables).

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8 35	6	10	9	35
Plant 2	9 10	12 20	13 20	7	50
Plant 3	14	9	16 10	5 30	40
DEMAND	45	20	30	30	125

u1 = 0

u1 + v1 = 8 yields v1 = 8

u2 + v1 = 9 yields u2 = 1

u2 + v2 = 12 yields v2 = 11

u2 + v3 = 13 yields v3 = 12

u3 + v3 = 16 yields u3 = 4

u3 + v4 = 5 yields v4 = 1

For each nonbasic variable, we now compute $\hat{c}_{ij} = u_i + v_j -$

c_{ij} $\hat{c}_{12} = 0 + 11 - 6 = 5$ $\hat{c}_{13} = 0 + 12 - 10 = 2$ $\hat{c}_{14} = 0 +$

$1 - 9 = -8$ $\hat{c}_{24} = 1 + 1 - 7 = -5$

$\hat{c}_{31} = 4 + 8 - 14 = -2$

$\hat{c}_{32} = 4 + 11 - 9 = 6$

Since $\hat{c}_{32}$ is the most positive one, we would next enter x_{32} into the basis: Each unit of x_{32} that is entered into the basis will decrease Powerco's cost by \$6.

The loop involving x_{32} is (3,2)-(3,3)-(2,3)-(2,2). = 10 (see table)

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8 35	6	10	9	35
Plant 2	9 10	12 **20–**	13 **20+**	7	50
Plant 3	14	9	16 **10–**	5 30	40
DEMAND	45	20	30	30	125

x_{33} would leave the basis. New bfs is shown at the following table:

u_i/v_j	8	11	12	7	SUPPLY
0	[8] 35	[6]	[10]	[9]	35
1	[9] 10	[12] 10	[13] 30	[7]	50
-2	[14]	[9] 10	[16]	[5] 30	40
DEMAND	45	20	30	30	125

$\hat{c}_{12} = 5$, $\hat{c}_{13} = 2$, $\hat{c}_{14} = -2$, $\hat{c}_{24} = 1$, $\hat{c}_{31} = -8$, $\hat{c}_{33} = -6$

Since $\hat{c}_{12}$ is the most positive one, we would next enter x_{12} into the basis.

The loop involving x_{12} is (1,2)-(2,2)-(2,1)-(1,1).   = 10 (see table)

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	[8] **35–**	[6]	[10]	[9]	35
Plant 2	[9] **10+**	[12] **10–**	[13] 30	[7]	50
Plant 3	[14]	[9] 10	[16]	[5] 30	40
DEMAND	45	20	30	30	125

x_{22} would leave the basis. New bfs is shown at the following table:

u_i/v_j	8	6	12	2	SUPPLY
0	[8] 25	[6] 10	[10]	[9]	35
1	[9] 20	[12]	[13] 30	[7]	50
3	[14]	[9] 10	[16]	[5] 30	40
DEMAND	45	20	30	30	125

$\hat{c}_{13} = 2$, $\hat{c}_{14} = -7$, $\hat{c}_{22} = -5$, $\hat{c}_{24} = -4$, $\hat{c}_{31} = -3$, $\hat{c}_{33} = -1$

Since $\hat{c}_{13}$ is the most positive one, we would next enter x_{13} into the basis.

The loop involving x_{13} is (1,3)-(2,3)-(2,1)-(1,1). = 25 (see table)

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8 **25–**	6 10	10	9	35
Plant 2	9 **20+**	12	13 **30–**	7	50
Plant 3	14	9 10	16	5 30	40
DEMAND	45	20	30	30	125

x_{11} would leave the basis. New bfs is shown at the following table:

u_i/v_j	6	6	10	2	SUPPLY
0	8	6 10	10 25	9	35
3	9 45	12	13 5	7	50
3	14	9 10	16	5 30	40
DEMAND	45	20	30	30	125

$\hat{c}_{11} = -2$, $\hat{c}_{14} = -7$, $\hat{c}_{22} = -3$, $\hat{c}_{24} = -2$, $\hat{c}_{31} = -5$, $\hat{c}_{33} = -3$

Since all $\hat{c}_{ij}$'s are negative, an optimal solution has been obtained.

Report

45 million kwh of electricity would be sent from plant 2 to city 1.

10 million kwh of electricity would be sent from plant 1 to city 2. Similarly, 10 million kwh of electricity would be sent from plant 3 to city 2.

25 million kwh of electricity would be sent from plant 1 to city 3. 5 million kwh of electricity would be sent from plant 2 to city 3.

30 million kwh of electricity would be sent from plant 3 to city 4
and Total shipping cost is:

$$z = .9\ (45) + 6\ (10) + 9\ (10) + 10\ (25) + 13\ (5) + 5\ (30) = \$\ 1020$$

4.4 TRANSSHIPMENT PROBLEMS

Sometimes a point in the shipment process can both receive goods from other points and send goods to other points. This point is called as **transhipment point** through which goods can be transhipped on their journey from a supply point to demand point.

Shipping problem with this characteristic is a transhipment problem.

The optimal solution to a transshipment problem can be found by converting this transshipment problem to a transportation problem and then solving this transportation problem.

Remark

As stated in “Formulating Transportation Problems”, we define a **supply point** to be a point that can send goods to another point but cannot receive goods from any other point.

Similarly, a **demand point** is a point that can receive goods from other points but cannot send goods to any other point.

Steps

1. If the problem is unbalanced, balance it

   Let *s* = total available supply (or demand) for balanced problem

2. Construct a transportation tableau as follows

   A row in the tableau will be needed for each supply point and transshipment point
   A column will be needed for each demand point and transshipment point

   Each supply point will have a supply equal to its original supply Each
   demand point will have a demand equal to its original demand

   Each transshipment point will have a supply equal to “that point’s original supply + *s*”

   Each transshipment point will have a demand equal to “that point’s original demand + *s*”

3. Solve the transportation problem

Example 1. Bosphorus

Bosphorus manufactures LCD TVs at two factories, one in Istanbul and one in Bruges. The Istanbul factory can produce up to 150 TVs per day, and the Bruges factory can produce up to 200 TVs per day. TVs are shipped by air to customers in London and Paris. The customers in each city require 130 TVs per day. Because of the deregulation of air fares, Bosphorus believes that it may be cheaper to first fly some TVs to Amsterdam or Munchen and then fly them to their final destinations.

The costs of flying a TV are shown at the table below. Bosphorus wants to minimize the total cost of shipping the required TVs to its customers.

€	To					
From	Istanbul	Bruges	Amsterdam	Munchen	London	Paris
Istanbul	0	-	8	13	25	28
Bruges	-	0	15	12	26	25
Amsterdam	-	-	0	6	16	17
Munchen	-	-	6	0	14	16
London	-	-	-	-	0	-
Paris	-	-	-	-	-	0

Answer:

In this problem Amsterdam and Munchen are *transshipment points*.

Step 1. Balancing the problem

Total supply = 150 + 200 = 350

Total demand = 130 + 130 = 260

Dummy's demand = 350 – 260 = 90

s = 350 (total available supply or demand for balanced problem) **Step 2.** Constructing a transportation tableau

Transshipment point's demand = Its original demand + *s* = 0 + 350 = 350

Transshipment point's supply = Its original supply + *s* = 0 + 350 = 350

	Amsterdam	Munchen	London	Paris	Dummy	Supply
Istanbul	8	13	25	28	0	150
Bruges	15	12	26	25	0	200
Amsterdam	0	6	16	17	0	350
Munchen	6	0	14	16	0	350
Demand	350	350	130	130	90	

Step 3. Solving the transportation problem

	Amsterdam	Munchen	London	Paris	Dummy	Supply
Istanbul	8 130	13	25	28	0 20	150
Bruges	15	12	26	25 130	0 70	200
Amsterdam	0 220	6	16 130	17	0	350
Munchen	6	0 350	14	16	0	350
Demand	350	350	130	130	90	1050

Report:

Bosphorus should produce 130 TVs at Istanbul, ship them to Amsterdam, and transship them from Amsterdam to London.

The 130 TVs produced at Bruges should be shipped directly to Paris. The total shipment is 6370 Euros.

4.5 ASSIGNMENT PROBLEMS

There is a special case of transportation problems where each supply point should be assigned to a demand point and each demand should be met. This certain class of problems is called as “assignment problems”. For example determining which employee or machine should be assigned to which job is an assignment problem.

4.5.1 LP Representation

An assignment problem is characterized by knowledge of the cost of assigning each supply point to each demand point: c_{ij}

On the other hand, a 0-1 integer variable x_{ij} is defined as follows

> x_{ij} = 1 if supply point i is assigned to meet the demands of demand point j
> x_{ij} = 0 if supply point i is not assigned to meet the demands of point j

In this case, the general LP representation of an assignment problem is

$$\min \sum_i \sum_j c_{ij} x_{ij}$$

$$\text{s.t.} \sum_j x_{ij} = 1 \qquad (i=1,2, ..., m) \qquad \text{Supply constraints}$$

$$\sum_i x_{ij} = 1 \qquad (j=1,2, ..., n) \qquad \text{Demand constraints}$$

$$x_{ij} = 0 \text{ or } x_{ij} = 1$$

4.5.2 Hungarian Method

Since all the supplies and demands for any assignment problem are integers, all variables in optimal solution of the problem must be integers. Since the RHS of each constraint is equal to 1, each x_{ij} must be a nonnegative integer that is no larger than 1, so each x_{ij} must equal 0 or 1.

Ignoring the x_{ij} = 0 or x_{ij} = 1 restrictions at the LP representation of the assignment problem, we see that we confront with a balanced transportation problem in which each supply point has a supply of 1 and each demand point has a demand of 1.

However, the high degree of degeneracy in an assignment problem may cause the Transportation Simplex to be an inefficient way of solving assignment problems.

For this reason and the fact that the algorithm is even simpler than the Transportation Simplex, the Hungarian method is usually used to solve assignment problems.

Remarks

1. To solve an assignment problem in which the goal is to maximize the objective function, multiply the profits matrix through by –1 and solve the problem as a **minimization** problem.
2. If the number of rows and columns in the cost matrix are unequal, the assignment problem is **unbalanced**. Any assignment problem should be balanced by the addition of one or more dummy points before it is solved by the Hungarian method.

Steps

1. Find the minimum cost each row of the $m*m$ cost matrix.
2. Construct a new matrix by subtracting from each cost the minimum cost in its row
3. For this new matrix, find the minimum cost in each column
4. Construct a new matrix (reduced cost matrix) by subtracting from each cost the minimum cost in its column
5. Draw the minimum number of lines (horizontal and/or vertical) that are needed to cover all the zeros in the reduced cost matrix. If m lines are required, an optimal solution is available among the covered zeros in the matrix. If fewer than m lines are needed, proceed to next step
6. Find the smallest cost (k) in the reduced cost matrix that is uncovered by the lines drawn in Step 5
7. Subtract k from each uncovered element of the reduced cost matrix and add k to each element that is covered by two lines. Return to Step 5

Example 1. Flight Crew

Four captain pilots (CP1, CP2, CP3, CP4) has evaluated four flight officers (FO1, FO2, FO3, FO4) according to perfection, adaptation, morale motivation in a 1-20 scale (1: very good, 20: very bad). Evaluation grades are given in the table. Flight

Company wants to assign each flight officer to a captain pilot according to these evaluations. Determine possible flight crews.

	FO1	FO2	FO3	FO4
CP1	2	4	6	10
CP2	2	12	6	5
CP3	7	8	3	9
CP4	14	5	8	7

Answer:

Step 1. For each row in the table we find the minimum cost: 2, 2, 3, and 5 respectively

Step 2 & 3. We subtract the row minimum from each cost in the row. For this new matrix, we find the minimum cost in each column

	0	2	4	8
	0	10	4	3
	4	5	0	6
	9	0	3	2
Column minimum	0	0	0	2

Step 4. We now subtract the column minimum from each cost in the column obtaining reduced cost matrix.

0	2	4	6
0	10	4	1
4	5	0	4
9	0	3	0

Step 5. As shown, lines through row 3, row 4, and column 1 cover all the zeros in the reduced cost matrix. The minimum number of lines for this operation is 3. Since fewer than four lines are required to cover all the zeros, solution is not optimal: we proceed to next step.

0	2	4	6
0	10	4	1
~~4~~	~~5~~	~~0~~	~~4~~
~~9~~	~~0~~	~~3~~	~~0~~

Step 6 & 7. The smallest uncovered cost equals 1. We now subtract 1 from each uncovered cost, add 1 to each twice-covered cost, and obtain

0	1	3	5
0	9	3	0
~~5~~	~~5~~	~~0~~	~~4~~
~~10~~	~~0~~	~~3~~	~~0~~

Four lines are now required to cover all the zeros: An optimal s9olution is available. Observe that the only covered 0 in column 3 is x_{33}, and in column 2 is x_{42}. As row 5 can not be used again, for column 4 the remaining zero is x_{24}. Finally we choose x_{11}.

Report:

CP1 should fly with FO1; CP2 should fly with FO4; CP3 should fly with FO3; and CP4 should fly with FO4.

Example 2. Maximization problem

	F	G	H	I	J
A	6	3	5	8	10
B	2	7	6	3	2
C	5	8	3	4	6
D	6	9	3	1	7
E	2	2	2	2	8

Report:

Optimal profit = 36

Assigments: A-I, B-H, C-G, D-F, E-J

Alternative optimal sol’n: A-I, B-H, C-F, D-G, E-J